Lectures on Quantum Field Theory

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Abstract

This is a lecture note prepared for the course on Quantum Field Theory held under Math4Phys program at IMB, université de Bourgogne.

Contents

Chapter 1

Introduction

1.1 Why QFT?

Q. What is QFT?

A. It is a universal framework to describe quantum many-body physics:

A practical method to describe many-body physics is the occupation number representation, which is namely the second quantization formalism. In the canonical quantization formalism, the field operator is constructed as superposition of (infinitely) many creation and annihilation operators.

Q. What does the universality mean?

A. The idea of the universality is as follows:

We are interested in various kinds of many-body interacting systems of elementary particles (quarks, electrons), nucleons, quasi-particles, macroscopic particles, etc. They actually have different microscopic description (Hamiltonian, Lagrangian), depending on a lot of parameters, e.g., mass parameters, coupling constants. However, in the low-energy regime, we may apply a unified description with very few parameters: Most of the microscopic parameters become irrelevant in that situation.:

Q. What is the unified description then?

A. It is sometimes called the (low-energy) effective theory [Wei79]:

In general, one can consider various forms of the interaction terms in the Hamiltonian or the Lagrangian. It is also possible to include derivatives, which are converted to multiplication of the momentum p. In the low-energy regime, we may put $p \ll 1$, so that we do not need to consider the higher derivative terms. Similarly, imposing a symmetry to the system, it also provides restrictions on the possible interactions. In this way, we may consider the effective field theory in the low-energy regime, which consists only of the appropriate degrees of freedom.

1.2 References

There are a lot of nice textbooks of QFT. Here is an incomplete list:

[Sre07] M. Srednicki, *Quantum Field Theory*, Cambridge Univ. Press, 2007.

[Nai05] V. P. Nair, *Quantum field theory: A modern perspective*, Graduate Texts in Contemporary Physics, Springer-Verlag, 2005.

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[Ryd96] L. H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1996.

[Wei95] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, 1995.

[PS95] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, USA, 1995.

[IZ80] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, International Series In Pure and Applied Physics, McGraw-Hill, New York, 1980.

1.3 Scalar field on a lattice

Let us demonstrate the idea of effective theory with an example, called the scalar field model on a lattice.[1] We define the scalar field defined on a D-dimensional lattice Λ^D,[2]

$$\phi(x) \in \mathbb{R}, \qquad x = (x_1, \ldots, x_D) \in \Lambda^D \subset \mathbb{R}^D. \tag{1.3.1}$$

Namely, the scalar field is interpreted as a map, $\phi : \Lambda^D \to \mathbb{R}$. See Sec. 2.1.1 for the definition of the field in general.

Let $a \in \mathbb{R}_{\geq 0}$ be the lattice spacing, then we may write $x = a \cdot n \in \Lambda^D$ where $n = (n_1, \ldots, n_D) \in \mathbb{Z}^D$. We impose the periodic boundary condition

$$x_\mu \simeq x_\mu + L = x_\mu + aN, \qquad L \in \mathbb{R}_{>0}, \quad N \in \mathbb{Z}_{>0}, \tag{1.3.2}$$

and thus the D-dimensional volume of the system is given by

$$\operatorname{vol} \Lambda^D = L^D = a^D N^D. \tag{1.3.3}$$

We consider the following Hamiltonian of the system:

$$H = \sum_{x \in \Lambda^D} \left[m_0^2 \, \phi(x)^2 + t \sum_{\mu=1}^{D} \phi(x + a\hat{\mu})\phi(x) + \sum_{k=2}^{\infty} \lambda_{2k} \, \phi(x)^{2k} \right], \tag{1.3.4}$$

$$=: H_0 + H_{\text{int}} \tag{1.3.5}$$

where we call the quadratic terms of the scalar field $\phi(x)$ the free part H_0, consisting of the (bare) mass term $m_0^2 \, \phi(x)^2$ and the hopping term $\phi(x + a\hat{\mu})\phi(x)$, with the hopping parameter

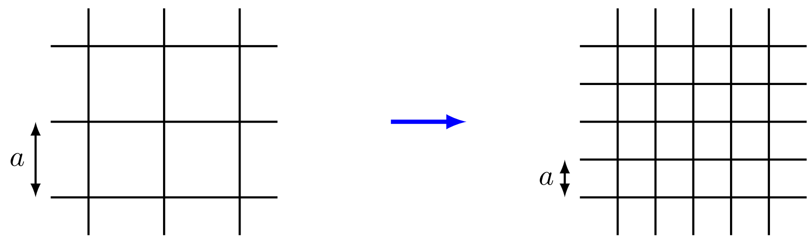
[1] One may refer to the textbooks about QFT on a lattice for details [Cre85, Rot12, Cre18].

[2] We can similarly discuss the complex scalar field $\phi(x) \in \mathbb{C}$, and also the multi-component field $\phi(x) \in \mathbb{R}^n$, $\mathbb{C}^n$.

$t \in \mathbb{R}$. We denote the unit vector in μ-direction by $\hat{\mu}$, so that the hopping term describes the nearest-neighbor propagation of the scalar field. The remaining terms are called the interaction part H_{int}. Although we formally consider infinitely many coupling constants $(\lambda_{2k})_{k\geq 1}$,[3] we typically consider finite number of the couplings in many cases.

1.3.1 Continuum limit $a \to 0$

Given such a lattice system, we are interested in the continuum limit, $a \to 0$, where we expect the universal behavior.



(1.3.6)

First of all, in this limit, the summation over the lattice is replaced with the integral:

$$\sum_{x\in\Lambda^D} a^D \longrightarrow \int \mathrm{d}^D x \, . \tag{1.3.7}$$

The factor a^D corresponds to the volume element of the present case, $\sum_{x\in\Lambda^D} a^D = \operatorname{vol}\Lambda^D$. In order to extract this volume factor, we should rescale the field and the parameters. For example, the mass term is rewritten as[4]

$$m_0^2\,\phi(x)^2 = a^D \left(\frac{m_0}{a}\right)^2 \left(\frac{\phi(x)}{a^{D/2-1}}\right)^2 , \tag{1.3.8}$$

so that we rescale the variables,

$$\frac{m_0}{a} \longrightarrow m_0 \, , \qquad \frac{\phi(x)}{a^{D/2-1}} \longrightarrow \phi(x) \, , \tag{1.3.9}$$

Namely, they have the following mass dimensions:

$$[m_0] = 1 \, , \qquad [\phi] = \frac{D}{2} - 1 \, . \tag{1.3.10}$$

From this point of view, if the parameter has a negative dimension, it does not contribute to the Hamiltonian (Lagrangian) in the continuum limit $a \to 0$, which is called **irrelevant**. On the other hand, it is called **relevant/marginal** if it has positive/zero dimension. We remark $[\phi] \geq 0$ for $D \geq 2$.

[3]For the moment, we do not consider the odd-power terms $\lambda_{2k+1}\phi(x)^{2k+1}$, which are not invariant under the transformation $\phi(x) \to -\phi(x)$. This is interpreted as the action of $\mathrm{O}(1) = \{\pm 1\} = \mathbb{Z}_2$. For the case of the complex scalar field, we similarly consider the Hamiltonian (Lagrangian), which is invariant under the U(1) transform (the phase rotation).

[4]This behavior is detemined to be consistent with the later argument.

1.3.2 Quadratic part

We apply the Taylor expansion to the kinetic term:

$$\phi(x + a\hat{\mu}) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \partial_\mu^n \phi(x) \,, \tag{1.3.11}$$

where we denote the partial derivative by $\partial_\mu^n = \dfrac{\partial^n}{\partial x_\mu^n}$. The quadratic part of the Hamiltonian is expanded by the lattice spacing a with the rescaled variables (1.3.9) as follows:

$$H_0 = \sum_{x \in \Lambda^D} a^D \left[(m_0^2 + D\frac{t}{a^2})\phi(x)^2 + t \sum_{\mu=1}^{D} \left(a^{-1}\, \partial_\mu \phi(x)\phi(x) + \frac{1}{2}\partial_\mu^2 \phi(x)\phi(x) \right) + O(a) \right] . \tag{1.3.12}$$

We remark that the $O(a^{-1})$ term does not contribute to the Hamiltonian since it is written as the total derivative, $\partial_\mu \phi(x)\phi(x) = \frac{1}{2}\partial_\mu(\phi(x)^2)$, and thus $\int \mathrm{d}x^D \, \partial_\mu(\phi(x)^2) = 0$ (Recall that we are now considering the periodic boundary condition; no boundary). Then, taking the continuum limit $a \to 0$, we can omit the irrelevant terms in $O(a)$, and obtain the following effective Hamiltonian:

$$\lim_{a \to 0} H_0 = \int \mathrm{d}^D x \left[m^2 \phi(x)^2 + \frac{1}{2} \partial_\mu^2 \phi(x)\phi(x) \right] = \int \mathrm{d}^D x \left[m^2 \phi(x)^2 - \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) \right] \tag{1.3.13}$$

where we put $t = 1$ for simplicity, and the new mass parameter is defined as[5]

$$m^2 = m_0^2 + \frac{D}{a^2} \,. \tag{1.3.14}$$

We again apply the integration by parts to obtain the second expression, $\partial_\mu^2 \phi(x)\phi(x) = \partial_\mu(\partial_\mu \phi(x)\phi(x)) - \partial_\mu \phi(x) \partial_\mu \phi(x)$.

1.3.3 Interaction part

Let us apply the same argument for the interaction part H_{int}. We rewrite

$$\lambda_{2k} \phi(x)^{2k} = a^D \frac{\lambda_{2k}}{a^{D-2k(D/2-1)}} \frac{\phi(x)^{2k}}{a^{2k(D/2-1)}} \tag{1.3.15}$$

then the coupling constant is rescaled as

$$\frac{\lambda_k}{a^{D-k(D/2-1)}} \longrightarrow \lambda_k \,, \qquad [\lambda_k] = D - k\left(\frac{D}{2} - 1\right) . \tag{1.3.16}$$

Now the dimension of the coupling constant can be both positive and negative, depending on the dimension D. Let us examine it in details.

$D = 2$

In this case, the scalar field is dimensionless, $[\phi] = 0$, and the coupling dimension does not depend on k, $[\lambda_k] = 2$ $(= D)$. Therefore, all the couplings are relevant, and we can consider various non-linear models in $D = 2$. See Sec. 2.4.4.

[5] Naively thinking, this new mass parameter involves a divergence due to the factor D/a^2 in the limit $a \to 0$. The idea of the **renormalization** is that such a divergence is cancelled with the bare mass m_0, so that the new parameter m could be finite. See Chapter 4 for details.

$D = 3$

The scalar field has the dimension $[\phi] = 1/2$, so that $[\lambda_k] = 3 - k/2$, which becomes non-negative only for $k \leq 6$. For $k > 6$, the coupling becomes irrelevant.

$D = 4$

In this case, the coupling at $k = 4$ is marginal, and the others are irrelevant.

$D \geq 5$

All the couplings $k \geq 4$ are irrelevant, so that we cannot consider the interacting field theory in the continuum limit.[6]

This argument says that, even if you start with the microscopic model with a lot of parameters, most of them become irrelevant which do not contribute in the continuum limit $a \to 0$. So, we can focus on the effective theory with a few relevant and marginal parameters. This is basically the idea of **renormalization group**, which will be discussed in Sec. 4.3. In this context, the relevant/irrelevant operators are called the renormalizable/non-renormalizable operators.

Exercise 1.1. *Consider the higher (next-nearest neighbor) hopping term, and the interaction term with a hopping,*

$$\sum_{1\leq\mu<\nu\leq D} \phi(x + a(\hat{\mu} + \hat{\nu}))\phi(x) \, , \qquad \sum_{\mu=1}^{D} \phi(x + a\hat{\mu})\phi(x)^3 \, , \tag{1.3.17}$$

then derive the effective Hamiltonian in the continuum limit $a \to 0$.

Exercise 1.2. *Consider the complex scalar system on the lattice, $\phi(x) \in \mathbb{C}$, with the Hamiltonian,*

$$H = \sum_{x\in\Lambda^D} \left[m_0^2 \, |\phi(x)|^2 + \frac{t}{2} \sum_{\mu=1}^{D} \Big(\phi(x + a\hat{\mu})^\dagger \phi(x) + \phi(x)^\dagger \phi(x + a\hat{\mu}) \Big) + \sum_{k=2}^{\infty} \lambda_{2k} |\phi(x)|^{2k} \right] , \tag{1.3.18}$$

then derive the effective Hamiltonian in the continuum limit $a \to 0$.

[6] For $D = 5, 6$, the coupling becomes relevant and marginal at $k = 3$. Hence, one can consider an interacting scalar theory having the cubic term $\phi(x)^3$.

Chapter 2

Symmetry and fields

In this Chapter, we start to discuss the Lorentz symmetry, which is an important symmetry for relativistic field theory. We then introduce the Lagrangian formalism, and discuss various field theories and their symmetries within the classical theory.

2.1 Lorentz symmetry

We consider the Lorentz symmetry, which is a fundamental symmetry of the relativistic field theory. In this note, we use the following convention for the $\mathsf{d} = D + 1$ dimensional Lorentzian spacetime metric

$$\eta_{\mu\nu} = \mathrm{diag}(+1, \underbrace{-1, \ldots, -1}_{D}) . \tag{2.1.1}$$

The manifold with this metric signature is called the Lorentzian manifold, which is a special case of the pseudo-Riemannian manifold with the signature $(\underbrace{+1, \ldots, +1}_{p}, \underbrace{-1, \ldots, -1}_{q})$. In the case with $q = 0$, it is called the Riemannian manifold. For the moment, we focus on the case $(p, q) = (1, D)$, which is the most relevant situation in the relativistic field theory.

We introduce the following convention for the vectors:

$$a^\mu = (a^0, a^1, \ldots, a^D) , \qquad a_\mu = \eta_{\mu\nu} a^\nu = (a^0, -a^1, \ldots, -a^D) . \tag{2.1.2}$$

The inner product of the vectors is written using the metric as follows:

$$a \cdot b = a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu = \eta^{\mu\nu} a_\mu b_\nu . \tag{2.1.3}$$

Let us consider the behavior under the transformation (a map $\Lambda : \mathbb{R}^{1,D} \to \mathbb{R}^{1,D}$),

$$\Lambda : \ x^\mu \ \longmapsto \ \Lambda^\mu{}_\nu x^\nu , \tag{2.1.4}$$

with

$$x \cdot x = x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu \ \overset{\Lambda}{\mapsto} \ \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma . \tag{2.1.5}$$

If the norm is invariant under the transform, we obtain

$$\begin{aligned} \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad &\Longleftrightarrow \quad \Lambda_\nu{}^\tau \Lambda^\nu{}_\sigma = \delta^\tau_\sigma \\ &\Longleftrightarrow \quad \Lambda_\nu{}^\tau = (\Lambda^{-1})^\tau{}_\nu = \eta^{\tau\sigma} (\Lambda^{-1})_\sigma{}^\rho \eta_{\rho\nu} \quad \Longleftrightarrow \quad \Lambda^{\mathrm{T}} = \eta \Lambda^{-1} \eta . \end{aligned} \tag{2.1.6}$$

Therefore, $\Lambda^{\mu}{}_{\nu}$ is an element of the orthogonal group $\mathrm{O}(1,D)$, which is known as the Lorentz group for $\mathsf{d} = D+1$ dimensional spacetime. It is also convenient to use the matrix notation to consider the inner product,

$$a \cdot b = (a^0, a^1, \ldots, a^D)\Big(\eta\Big)\begin{pmatrix} b^0 \\ \vdots \\ b^D \end{pmatrix} = \mathrm{tr}\left[\eta \begin{pmatrix} b^0 \\ \vdots \\ b^D \end{pmatrix}(a^0, a^1, \ldots, a^D)\right], \tag{2.1.7}$$

which is analogous to the cyclic property of the trace. In this notation, the Lorentz transformation of the norm behaves as

$$(\Lambda x)\cdot(\Lambda x) = \mathrm{tr}\left[\eta\Lambda \begin{pmatrix} x^0 \\ \vdots \\ x^D \end{pmatrix}(x^0, x^1, \ldots, x^D)\Lambda^{\mathrm{T}}\right] = \mathrm{tr}\left[\Lambda^{\mathrm{T}}\eta\Lambda \begin{pmatrix} x^0 \\ \vdots \\ x^D \end{pmatrix}(x^0, x^1, \ldots, x^D)\right]. \tag{2.1.8}$$

Hence, we obtain the condition, $\eta = \Lambda^{\mathrm{T}}\eta\Lambda \iff \Lambda^{\mathrm{T}} = \eta\Lambda^{-1}\eta$.

There are four connected parts for $\mathrm{O}(1,D)$, which are related through the parity and the time-reversal operations, $\mathrm{P} = \mathrm{diag}(+1,-1,\ldots,-1)$, and $\mathrm{T} = \mathrm{diag}(-1,+1,\ldots,+1)$.[1] The connected part, which contains the identity $\mathbb{1} = \mathrm{diag}(+1,\ldots,+1)$, is called the restricted Lorentz group denoted by $\mathrm{SO}^+(1,D)$ or simply $\mathrm{SO}(1,D)$ if no confusion.

Exercise 2.1 (Isomorphisms of the spin group)**.** *The double cover of* $\mathrm{SO}(n)$ *is called the* ***spin group*** *denoted by* $\mathrm{Spin}(n)$*. Show the following isomorphisms for the spin group:*

$$\mathrm{Spin}(1,1) = \mathrm{GL}(1,\mathbb{R}), \qquad \mathrm{Spin}(1,2) = \mathrm{SL}(2,\mathbb{R}), \qquad \mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C}). \tag{2.1.9}$$

Then, discuss the isofmorphism by taking the $\mathbb{Z}_2$*-quotient*

$$\mathrm{SO}^+(1,3) = \mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \qquad \textit{with} \qquad \mathbb{Z}_2 = \{\pm 1\}. \tag{2.1.10}$$

2.1.1 Representation and fields

Let $\varphi(x) \in V$ be a field taking a value in the vector space V, e.g., $V = \mathbb{R}^n$, $\mathbb{C}^n$. Namely, the field defines a map $\varphi : M \longrightarrow V$, where M is a (base) manifold on which the field is defined, e.g., $M = \mathbb{R}^{1,D}$. Under the Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \qquad \Lambda \in \mathrm{SO}(1,D), \tag{2.1.11}$$

the field $\varphi(x)$ transforms as follows in general:

$$\varphi'_i(x') = \rho(\Lambda)_i{}^j \varphi_j(x), \tag{2.1.12}$$

where $\rho(\Lambda) \in \mathrm{GL}(V)$ (e.g., $\mathrm{GL}(n,\mathbb{C})$ for $V = \mathbb{C}^n$) obeying the relation,

$$\rho(\Lambda_1\Lambda_2) = \rho(\Lambda_1)\rho(\Lambda_2) \qquad \text{for} \qquad \forall \Lambda_1, \Lambda_2 \in \mathrm{SO}(1,D). \tag{2.1.13}$$

In other words, it provides a map $\rho : \mathrm{SO}(1,D) \to \mathrm{GL}(V)$, which defines the **representation of the Lorentz group**. The vector space V is called the representation space in this context (and also the target space. See Sec. 2.4.4). From this point of view, we can classify the fields based on the representation theory of the Lorentz group.

[1] In fact, the parity matrix P coincides with the Lorentzian metric η under the notation $\eta = (+ - --)$. Therefore, the relation discussed in (2.1.6) is understood as $\Lambda^{-1} = \eta\Lambda^{\mathrm{T}}\eta = \mathrm{P}\Lambda^{\mathrm{T}}\mathrm{P}$. Under the different notation $\eta = (- + ++)$, the Lorentz metric coincides with T.

$\mathsf{d} = 3 + 1$

Let us focus on an important example, $\mathsf{d} = 3 + 1$. In this case, the Lorentz group is given by $\mathrm{SO}(1, 3)$, which is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. The Lorentz group $\mathrm{SO}(1, 3)$ is generated by $\binom{4}{2} = 6$ "rotations" $(M_{\mu\nu})_{\mu,\nu=0,\ldots,3}$, which are antisymmetric $M_{\mu\nu} = -M_{\nu\mu}$, and split into three spatial rotations $J = (J_i = \frac{1}{2}\epsilon_{ijk}M^{jk})_{i=1,2,3}$ and three temporal ones, called the boosts $K = (K_i = M_{i0} = -M_{0i})_{i=1,2,3}$.

In order to see the underlying algebraic structure, we consider the infinitesimal version of the Lorentz transformation with the generators,

$$\Lambda^{\mu}{}_{\nu} = \exp\left(-\frac{\mathrm{i}}{2}\epsilon^{\rho\sigma}M_{\rho\sigma}\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{\mathrm{i}}{2}\epsilon^{\rho\sigma}(M_{\rho\sigma})^{\mu}{}_{\nu} + O(\epsilon^2)\,, \tag{2.1.14}$$

with the rank 2 antisymmetric tensor $\epsilon^{\rho\sigma} = -\epsilon^{\sigma\rho}$. Identifying this with $\delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}$, we obtain

$$(M_{\rho\sigma})^{\mu}{}_{\nu} = \mathrm{i}\left(\delta^{\mu}_{\rho}\eta_{\sigma\nu} - \delta^{\mu}_{\sigma}\eta_{\rho\nu}\right)\,, \tag{2.1.15}$$

which obeys the commutation relation

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})\,. \tag{2.1.16}$$

From this relation, we obtain

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k\,, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k\,, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k \tag{2.1.17}$$

Combining them as

$$A = \frac{1}{2}(J + \mathrm{i}K)\,, \qquad B = \frac{1}{2}(J - \mathrm{i}K)\,, \tag{2.1.18}$$

we see that $A = (A_i)_{i=1,2,3}$ and $B = (B_i)_{i=1,2,3}$ separately obey the commutation relations of the Lie algebra $\mathfrak{su}(2)$, whose complexification provides $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$,

$$[A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k\,, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k\,, \qquad [A_i, B_j] = 0\,. \tag{2.1.19}$$

This implies that the representation of the Lorentz group is characterized by that of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ labeled by a pair of half-integers, $(n, m) \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Here, the dimension of the representation (n, m) coincides with that for the representation space V, $\dim V = (2n + 1)(2m + 1)$, and the quantity $s = n + m$ is called the **spin**. We show several examples in the following:

Field	Representation	Dimension	Spin
scalar	(0,0)	1	0
spinor	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$2 + 2$	$\frac{1}{2}$
vector	$(\frac{1}{2}, \frac{1}{2})$	4	1
((anti-)selfdual) tensor	$(1, 0) \oplus (0, 1)$	$3 + 3$	1

(2.1.20)

It is also possible to consider higher-spin fields, Rarita–Schwinger field and gravitino for spin-$\frac{3}{2}$, and graviton for spin-2. It is known that the bosonic/fermionic field has integer/half-integer spin (a.k.a., the spin-statistics theorem).

Spinor field

Let us discuss the spinor field in details. For example, the $(0, \frac{1}{2})$-representation of the Lorentz group, corresponding to the two-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ denoted by $\mathbf{2}$-representation, is given as follows:[2]

$$\mathbf{2} = \left(0, \frac{1}{2}\right) : \qquad \rho(A_i) = 0\,, \qquad \rho(B_i) = \frac{1}{2}\sigma_i\,, \qquad V = \mathbb{C}^2\,, \tag{2.1.21}$$

where $(\sigma_i)_{i=1,2,3}$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.22}$$

Namely, we have $\rho(J_i) = \frac{1}{2}\sigma_i$, $\rho(K_i) = \frac{\mathrm{i}}{2}\sigma_i$ for the $\mathbf{2}$-representation. The other representation $(\frac{1}{2}, 0)$ is given by the complex conjugation, denoted by the $\mathbf{2}^*$-representation, as $\rho(J_i) = \frac{1}{2}\sigma_i^*$, $\rho(K_i) = -\frac{\mathrm{i}}{2}\sigma_i^*$, so that we have

$$\mathbf{2}^* = \left(\frac{1}{2}, 0\right) : \qquad \rho(A_i) = \frac{1}{2}\sigma_i^*\,, \qquad \rho(B_i) = 0\,, \qquad V = \mathbb{C}^2\,. \tag{2.1.23}$$

We denote the spinor fields associated with the representations, $\mathbf{2}$ and $\mathbf{2}^*$, by $\xi = (\xi_\alpha)_{\alpha=1,2} \in V$ and $\eta = (\eta_{\dot\alpha})_{\dot\alpha=1,2} \in V$. We here distinguish the indices α and $\dot\alpha$ associated with $\mathbf{2}$ and $\mathbf{2}^*$.

The contraction of the indices is taken with the invariant tensor of $\mathrm{SL}(2,\mathbb{C})$,

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1.24}$$

For example, the Lorentz scalar is given by

$$\epsilon^{\alpha\beta}\xi_\beta\eta_\alpha = \xi^\alpha\eta_\alpha = -\xi_\beta\eta^\beta\,, \qquad \epsilon^{\dot\alpha\dot\beta}\xi_{\dot\beta}\eta_{\dot\alpha} = \xi^{\dot\alpha}\eta_{\dot\alpha} = -\xi_{\dot\beta}\eta^{\dot\beta}\,. \tag{2.1.25}$$

We remark $\epsilon^{\mathrm{T}} = \epsilon^{-1} = -\epsilon$. Then, the vector is constructed by $\mathbf{2}$ and $\mathbf{2}^*$ as follows:

$$X_\mu = \xi^\alpha(\sigma_\mu)_{\alpha\dot\beta}\eta^{\dot\beta} \tag{2.1.26}$$

where we define

$$(\sigma_\mu)_{\alpha\dot\beta} = (1, \sigma_{i=1,2,3})_{\alpha\dot\beta}\,, \qquad (\bar\sigma_\mu)^{\dot\alpha\beta} = (1, -\sigma_{i=1,2,3})^{\dot\alpha\beta}\,. \tag{2.1.27}$$

Using the Pauli matrices, it is possible to convert the vector to the mixed-spinor,

$$X^\mu(\sigma_\mu)_{\alpha\dot\beta} =: X_{\alpha\dot\beta}\,. \tag{2.1.28}$$

We also define

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{\mathrm{i}}{2}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_\alpha^{\ \beta}\,, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{\mathrm{i}}{2}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}_{\ \dot\beta}\,, \tag{2.1.29}$$

[2] We again remark that A and B obey $\mathfrak{su}(2)$ Lie algebra relation (2.1.19), and thus $\rho(A)$ and $\rho(B)$ are now representations of $\mathfrak{su}(2)$ Lie algebra. Hence, they correspond to the trivial representation $U_A = \exp(\mathrm{i}\epsilon^i\rho(A_i)) = 1$ and the two-dimensional representation $U_B = \exp(\mathrm{i}\epsilon^i\rho(B_i))$ of $\mathrm{SU}(2)$. In fact, although $\mathrm{SL}(2)$ and $\mathrm{SU}(2)$ are different Lie groups, finite dimensional irreducible representations of $\mathrm{SL}(2)$ are equivalent to unitary irreducible representations of $\mathrm{SU}(2)$ (unitarian trick).

which are (anti-)selfdual tensors,

$$\sigma^{\mu\nu} = +\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,, \qquad \bar{\sigma}^{\mu\nu} = -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}\,. \tag{2.1.30}$$

We remark that any anti-symmetric tensor (six components) splits into selfdual and anti-selfdual tensors (three components for each).

These constructions are based on the tensor product of **2** and $\mathbf{2}^*$ representations,

$$\mathbf{2}\otimes\mathbf{2} = \left(0,\frac{1}{2}\right)\otimes\left(0,\frac{1}{2}\right) = (0,0)\oplus(0,1)\,, \tag{2.1.31a}$$

$$\mathbf{2}^*\otimes\mathbf{2}^* = \left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right) = (0,0)\oplus(1,0)\,, \tag{2.1.31b}$$

$$\mathbf{2}\otimes\mathbf{2}^* = \left(0,\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right) = \left(\frac{1}{2},\frac{1}{2}\right). \tag{2.1.31c}$$

See also the table of the Lorentz group representations (2.1.20).

2.2 Lagrangian formalism

Let us introduce the Lagrangian formalism for field theories. We here consider the Lagrangian with the following conditions:

1. Locality

   The action S is given by the integral of the Lagrangian density which consists of the field $\varphi(x)$ and its derivatives $\partial\varphi(x)$,

   $$S[\varphi] = \int \mathrm{d}t \int \mathrm{d}^D x\, \mathscr{L}(\varphi(x),\partial\varphi(x)) = \int \mathrm{d}^{\mathsf{d}} x\, \mathscr{L}(\varphi(x),\partial\varphi(x))\,, \tag{2.2.1}$$

   where $\mathsf{d} = D+1$ is the spacetime dimension.

2. Unitarity

   The Lagrangian density is a real number in the classical theory, and a hermitian operator in the quantum theory, to guarantee the probability conservation.

3. Poincaré invariance

   The Lagrangian density is invariant under the Lorentz transformation and the translation. We also require the discrete C, P, and T symmetries (but not always).

4. In order that the resulting equation of motion becomes a second order differential equation, the Lagrangian does not contain higher derivative terms. (Otherwise, it may violate the causality.)

2.3 Symmetry and conservation law

We discuss the role of symmetry in the Lagrangian formalism. In particular, we show that one can construct the conserved quantity if there exists a continuous symmetry (parametrized by a continuous parameter).

2.3.1 Euler–Lagrange equation

We start with the Lagrangian (2.2.1), then take the variation with the field $\varphi(x) \to \varphi(x)+\delta\varphi(x)$,

$$\begin{aligned} \delta S &= \int \mathrm{d}^\mathsf{d} x \left(\frac{\partial \mathscr{L}}{\partial \varphi(x)} \delta\varphi(x) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \delta \partial_\mu \varphi(x) \right) \\ &= \int \mathrm{d}^\mathsf{d} x \left(\frac{\partial \mathscr{L}}{\partial \varphi(x)} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \right) \right) \delta\varphi(x) + \int \mathrm{d}^\mathsf{d} x \, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \delta\varphi(x) \right) . \end{aligned} \tag{2.3.1}$$

Assuming that the total derivative term does not contribute here, we obtain the Euler–Lagrange equation (or simply called the equation of motion) from the stationary condition under the variation,

$$0 = \frac{\delta S}{\delta \varphi(x)} \quad \Longrightarrow \quad 0 = \frac{\partial \mathscr{L}}{\partial \varphi(x)} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \right) . \tag{2.3.2}$$

2.3.2 Noether's theorem

Assume that the action S is invariant under the infinitesimal transformation of the field,

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \epsilon \, G(\varphi(x)) \, , \tag{2.3.3}$$

where ϵ is an infinitesimal parameter. We also denote $G(\varphi(x)) = \delta\varphi(x)$. When ϵ is a constant, the corresponding symmetry is called **global symmetry**, while it is called **local symmetry**, when ϵ depends on a point x. We focus on the global symmetry for the moment. In this case, the variation of the Lagrangian is given by

$$\begin{aligned} \delta \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial \varphi(x)} \epsilon G(\varphi(x)) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \partial_\mu (\epsilon G(\varphi(x))) \\ &= \epsilon \left(\frac{\partial \mathscr{L}}{\partial \varphi(x)} G(\varphi(x)) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \partial_\mu G(\varphi(x)) \right) \\ &\overset{(2.3.2)}{=} \epsilon \left(\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \right) G(\varphi(x)) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \partial_\mu G(\varphi(x)) \right) \\ &= \epsilon \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} G(\varphi(x)) \right) . \end{aligned} \tag{2.3.4}$$

In order that the action is invariant under the variation, this should be written as a total derivative,

$$\epsilon \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} G(\varphi(x)) \right) = \epsilon \partial_\mu X^\mu (\varphi(x)) \, , \tag{2.3.5}$$

which implies the conservation law,

$$\partial_\mu j^\mu(x) = 0 \tag{2.3.6}$$

of the current, which is called **Noether current**,

$$j^\mu(x) = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} G(\varphi(x)) - X^\mu(\varphi(x)) \, . \tag{2.3.7}$$

Noether's theorem states the existence of the Noether current for the system having a continuous global symmetry.

Noether charge

From the conserved current, we can construct the Noether charge (also called the conserved charge)

$$Q = \int \mathrm{d}^D x \, j^0(x) \quad \Longrightarrow \quad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \,. \tag{2.3.8}$$

Define the unitary operator

$$U_\epsilon = \exp\left(\mathrm{i}\epsilon Q\right) , \tag{2.3.9}$$

then we obtain

$$R_\epsilon \cdot \varphi(x) := U_\epsilon \varphi(x) U_\epsilon^{-1} = \varphi(x) + \epsilon[\mathrm{i}Q, \varphi(x)] + O(\epsilon^2) \,. \tag{2.3.10}$$

Therefore, the Noether charge plays a role of the generator of the infinitesimal transformation,

$$[\mathrm{i}Q, \varphi(x)] = G(\varphi(x)) \,. \tag{2.3.11}$$

We remark that the Noether current is not unique, and it is still conserved after modification by the anti-symmetric tensor,

$$j^\mu(x) \longrightarrow j^\mu(x) + \partial_\nu f^{\mu\nu}(x) \,, \qquad f^{\mu\nu}(x) = -f^{\nu\mu}(x) \tag{2.3.12}$$

because[3]

$$\partial_\mu \partial_\nu f^{\mu\nu}(x) = 0 \,. \tag{2.3.13}$$

The Noether charge Q is not affected by this modification, as long as there is no surface contribution,

$$\int \mathrm{d}^D x \, \partial_i f^{0i}(x) = 0 \,. \tag{2.3.14}$$

Energy-momentum tensor

We consider the translation[4]

$$x^\nu \longrightarrow x'^\nu = x^\nu - \epsilon^\nu \,. \tag{2.3.15}$$

Therefore, we obtain

$$\varphi'(x') = \varphi(x) \quad \Longrightarrow \quad \varphi'(x) = \varphi'(x' + \epsilon) \approx \varphi(x) + \epsilon^\nu \partial_\nu \varphi(x) \,. \tag{2.3.16}$$

Similarly, the Lagrangian density behaves

$$\mathscr{L}(\varphi'(x), \partial\varphi'(x)) \approx \mathscr{L}(\varphi(x), \partial\varphi(x)) + \epsilon^\nu \partial_\nu \mathscr{L}(\varphi(x), \partial\varphi(x)) \,. \tag{2.3.17}$$

Comparing with the previous argument, we obtain the **energy-momentum tensor**,

$$T^\mu_{\ \nu}(x) = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi(x))} \partial_\nu \varphi(x) - \delta^\mu_\nu \mathscr{L}(\varphi(x), \partial\varphi(x)) \,, \tag{2.3.18}$$

with the conservation law, $\partial_\mu T^\mu_{\ \nu} = 0$, and the Noether charge

$$P_\nu = \int \mathrm{d}^D x \, T^0_{\ \nu}(x) \,, \tag{2.3.19}$$

which is a generator of the translation.

[3] If the function $f^{\mu\nu}(x)$ is a smooth and single-valued function.

[4] Since the translation is parametrized by a vector ϵ^ν, it is isomorphic to $\mathbb{R}^{1,D}$. Combining the translation symmetry, the Lorentz group is promoted to a non-compact group, called the **Poincaré group**.

Generalized global symmetry

Using the differential form notation, the conservation low of the current (2.3.6) can be written as follows,

$$d \star j = 0\,, \tag{2.3.20}$$

where the current j is interpreted as a one-form, d is the exterior derivative, and $\star$ is the Hodge dual operator with respect to $\mathsf{d} = D + 1$ dimensional manifold, so that the dual current $\star j$ is a D-form.[5] In this formulation, the Noether charge (2.3.8) is given by

$$Q = \int_{M_D} \star j\,, \tag{2.3.21}$$

where M_D can be taken to be a general D-dimensional submanifold of the spacetime.

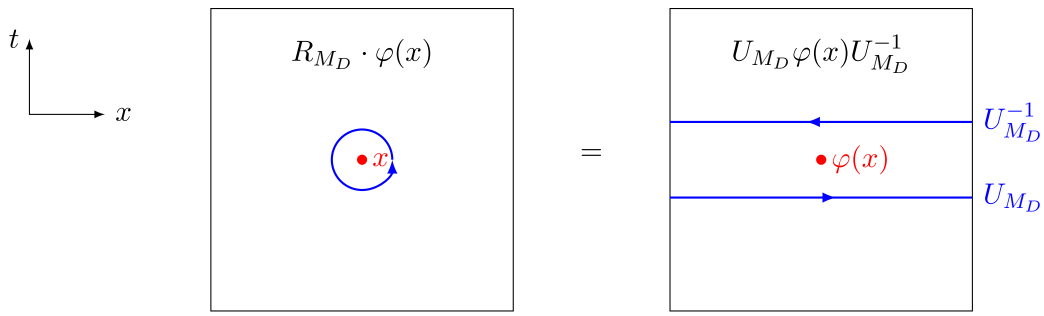
We then define the symmetry generator (analog of the unitary operator (2.3.9)) associated with M_D,

$$U_{M_D} = \exp\left(\mathrm{i}Q\right) = \exp\left(\mathrm{i}\int_{M_D} \star j\right). \tag{2.3.22}$$

The action of this generator is similarly defined as before (2.3.10),

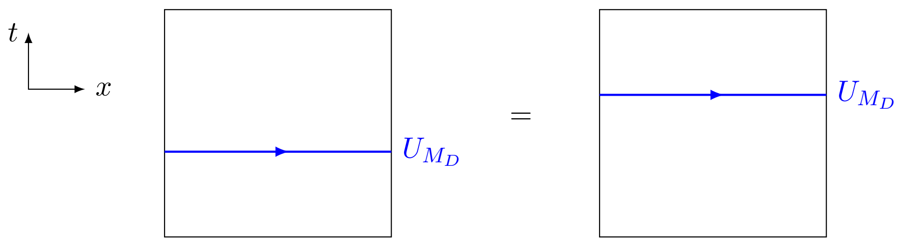
$$R_{M_D} \cdot \varphi(x) := U_{M_D} \varphi(x)\, U_{M_D}^{-1}\,. \tag{2.3.23}$$

We consider the case $D = 1$ and $M_D = \mathbb{R}$ (x-direction) for simplicity. In this case, the action of the generator (2.3.23) is graphically understood as follows,



(2.3.24)

From this point of view, this operation is topological in the sense that the operator R_{M_D} topologically wraps the field $\varphi(x)$. We remark that, due to the conservation law, the generator is translation invariant in t-direction, $\dfrac{\mathrm{d}U_{M_D}}{\mathrm{d}t} = 0$,



(2.3.25)

One can generalize this argument as follows. We start with a $(p+1)$-form conserved current j_{p+1} and its dual $\star j_{p+1}$, which is a $D - p$ form. The conservation law is similarly written as

$$d \star j_{p+1} = 0\,. \tag{2.3.26}$$

[5] We will discuss the differential form in Sec. 2.6.3 for more details.

Then, we define the symmetry generator associated with a $(D-p)$-dimensional submanifold M_{D-p},

$$U_{M_{D-p}} = \exp\left(\mathrm{i}\int_{M_{D-p}} \star j_p\right). \tag{2.3.27}$$

In order to topologically wrap the field (the charged object under the symmetry generator), it should be extended in a p-dimensional submanifold M_p, which defines the action of the generator,

$$R_{M_{D-p}} \cdot \varphi(M_p) := U_{M_{D-p}} \varphi(M_p)\, U_{M_{D-p}}^{-1}. \tag{2.3.28}$$

The global symmetry of such an extended (namely non-local) object is called the **generalized global symmetry** [GKSW14]. In particular, the symmetry for a p-dimensionally extended object is called the **p-form symmetry**, where the ordinary local symmetry corresponds to the 0-form symmetry.

2.3.3 Conformal symmetry

The Lorentz rotation and the translation are fundamental spacetime symmetries in the relativistic field theory. In addition, one may consider a special class of spacetime symmetries as follows:

$$\text{Dilatation (scale transform)} : x^\mu \longrightarrow x'^\mu = \mathrm{e}^{-\alpha} x^\mu = x^\mu - \alpha x^\mu + \cdots \tag{2.3.29a}$$

$$\text{Special conformal transform} : x^\mu \longrightarrow x'^\mu = \frac{x^\mu - \beta^\mu x^2}{1 - 2b\cdot x + b^2 x^2} = x^\mu - \beta^\mu x^2 - 2\beta^\nu x_\nu x^\mu + \cdots \tag{2.3.29b}$$

Exercise 2.2 (Special conformal transform)**.** *Verify that the special conformal transform* (2.3.29b) *can be rewritten in the following form,*

$$\frac{x^\mu}{x^2} \longrightarrow \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - \beta^\mu. \tag{2.3.30}$$

Exercise 2.3 (Dilaton current)**.** *Let D^μ be the Noether current associated with the dilatation. It is known that one can modify the energy-momentum tensor using a proper anti-symmetric tensor* (2.3.12)*, such that we have the relation*[6]

$$D^\mu = x^\nu T^\mu{}_\nu. \tag{2.3.31}$$

Show that the divergence of the dilaton current is given by the trace of the energy-momentum tensor,

$$\partial_\mu D^\mu = T^\mu{}_\mu. \tag{2.3.32}$$

Similarly to the translation, the dilatation and the special conformal transformation are non-compact, and isomorphic to $\mathbb{R}$ and $\mathbb{R}^{1,D}$. All these transformations are combined into the **conformal group**, which is given by $\mathrm{SO}(2, D+1)$,[7] and the theory having this conformal

[6]We remark that, after this modification, the energy-momentum tensor is not symmetric in general.

[7]We remark that $\mathrm{SO}(2, D+1)$ is the isometry group of the $(D+2)$-dimensional **Anti-de Sitter (AdS) space**, denoted by AdS_{D+2}. This agreement is one of the aspects of the **AdS/CFT correspondence**, which is the correspondence between $(D+2)$-dimensional AdS space and $(D+1)$-dimensional CFT. See, for example, [AE15, BBS07] for details.

symmetry is called the **conformal field theory (CFT)**:

$$\left.\begin{array}{ll} \left.\begin{array}{ll} \text{Lorentz group} & \mathrm{SO}(1,D) \\ \text{Translation} & \mathbb{R}^{1,D} \end{array}\right\} \text{Poincaré} & \\ \begin{array}{ll} \text{Dilatation} & \mathbb{R} \\ \text{Special conformal} & \mathbb{R}^{1,D} \end{array} & \end{array}\right\} \begin{array}{c}\text{Conformal group} \\ \mathrm{SO}(2,D+1)\end{array} \tag{2.3.33}$$

Exercise 2.4 (Counting the dimensions)**.** *The dimension of the orthogonal group* $\mathrm{SO}(p,q)$ *is given by*

$$\dim \mathrm{SO}(p,q) = \frac{1}{2}(p+q)(p+q-1)\,. \tag{2.3.34}$$

Verify that

$$\dim \mathrm{SO}(1,D) + \dim \mathbb{R}^{1,D} + \dim \mathbb{R} + \dim \mathbb{R}^{1,D} = \dim \mathrm{SO}(2,D+1)\,. \tag{2.3.35}$$

As seen in Exercise 2.1, the double cover of the conformal group is given by the (non-compact) spin group $\mathrm{Spin}(2,D+1)$, which show the following isomorphisms:

$$\mathrm{Spin}(2,2) = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\,, \quad \mathrm{Spin}(2,3) = \mathrm{Sp}(4,\mathbb{R})\,, \quad \mathrm{Spin}(2,4) = \mathrm{SU}(2,2)\,. \tag{2.3.36}$$

We remark that the conformal symmetry in the case $D = 1$ ($\mathsf{d} = 2$) is special: In this case, an arbitrary holomorphic function generates the conformal transformation, and therefore the conformal symmetry is enhanced to infinite dimensional symmetry. The symmetry algebra describing the two-dimensional conformal symmetry is called the **Virasoro algebra**, which is an infinite dimensional Lie algebra characterized by the following algebraic relation,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\,\delta_{n+m,0}\,, \qquad n,m \in \mathbb{Z}\,. \tag{2.3.37}$$

The element c is the **central charge**, which commutes with all other generators $(L_n)_{n\in\mathbb{Z}}$. Hence, one can treat it a (complex) number. See, for example, [DFMS97, Gab99, Sch05, Car08, Rib14, Tes17] for details.

Exercise 2.5 ($\mathfrak{sl}_2$ subalgebra of the Virasoro algebra)**.** *Verify that the generators* $(L_0, L_{\pm 1})$ *of the Virasoro algebra form the* $\mathfrak{sl}_2$ *subalgebra.*

In two dimensions, the holomorphic and anti-holomorphic sectors can be independently discussed, and one can consider the Virasoro algebra for each sector. In fact, as seen from the isomorphism (2.3.36), the two-dimensional conformal group consists of two $\mathrm{SL}(2)$ groups, which are interpreted as the subgroups of the infinite conformal symmetries associated with the holomorphic and anti-holomorphic sectors.

2.4 Scalar field

2.4.1 Real scalar field

Let us consider the scalar field theories. The simplest one is the real scalar field, $\phi(x) \in \mathbb{R}$, with the following Lagrangian:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,. \tag{2.4.1}$$

This model has a discrete symmetry $\mathbb{Z}_2 = \mathrm{O}(1)$, $\phi(x) \leftrightarrow -\phi(x)$, but does not have any continuous symmetry, so that we cannot discuss the Noether current in this case.

The corresponding action is given by

$$\begin{aligned} S &= \int \mathrm{d}^{\mathsf{d}}x \, \mathscr{L}(\phi, \partial\phi) \\ &= \int \mathrm{d}^{\mathsf{d}}x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right) \\ &= \int \mathrm{d}^{\mathsf{d}}x \left(-\frac{1}{2}\phi\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right) + \int \mathrm{d}^{\mathsf{d}}x \, \frac{1}{2}\partial_\mu(\phi\partial^\mu\phi) \, . \end{aligned} \tag{2.4.2}$$

We assume that the last term does not contribute to the action since it just gives the surface term.

Exercise 2.6 (Klein–Gordon equation)**.** *Derive the Euler–Lagrange equation for this Lagrangian:*

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi + \frac{\lambda}{3!}\phi^3 = 0 \, . \tag{2.4.3}$$

This is actually a non-linear wave equation due to the cubic term ϕ^3. When $\lambda = 0$, it is reduced to a linear wave equation, a.k.a., ***Klein–Gordon equation****.*

In this case, the mass dimensions of the field and the parameters as follows:

$$[m] = 1 \, , \qquad [\phi] = \frac{\mathsf{d}}{2} - 1 \, , \qquad [\lambda] = 4 - \mathsf{d} \, . \tag{2.4.4}$$

We remark that the dimension of the field ϕ is modified from (1.3.10). This is related to the correspondence between D-dimensional statistical mechanics (statistical field theory) and $\mathsf{d}(= D+1)$-dimensional quantum field theory. In $\mathsf{d} = 4$, in particular, the coupling constant of the higher term, $\phi(x)^k$ with $k > 4$, has a negative dimension, which is irrelevant.

2.4.2 Complex scalar field

The next example is the complex scalar field theory,

$$\mathscr{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2 \, . \tag{2.4.5}$$

In this case, the Lagrangian is invariant under the U(1) transformation,

$$\phi(x) \longrightarrow \mathrm{e}^{\mathrm{i}\theta}\phi(x) = \phi(x) + \mathrm{i}\theta\phi(x) + O(\theta^2) \, , \tag{2.4.6a}$$

$$\phi^*(x) \longrightarrow \mathrm{e}^{-\mathrm{i}\theta}\phi^*(x) = \phi^*(x) - \mathrm{i}\theta\phi^*(x) + O(\theta^2) \, . \tag{2.4.6b}$$

Applying the Noether current formula (2.3.7), we obtain the U(1) current,[8]

$$j^\mu = \mathrm{i}\phi\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} - \mathrm{i}\phi^*\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^*)} = \mathrm{i}\partial^\mu\phi^*\phi - \mathrm{i}\phi^*\partial^\mu\phi \, , \tag{2.4.7}$$

with the conserved charge,

$$Q = \int \mathrm{d}^D x \, j^0 = \int \mathrm{d}^D x \, \mathrm{i}\left(\partial^0\phi^*\phi - \phi^*\partial^0\phi\right) \tag{2.4.8}$$

Exercise 2.7 (Klein–Gordon equation for the complex field)**.** *Derive the Euler–Lagrange equation for the complex scalar field $\phi(x)$:*

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi + \lambda(\phi^*\phi)\phi = 0 \, , \tag{2.4.9}$$

and similarly for $\phi^*(x)$.

[8] See the Maurer–Cartan form discussed in Sec. 2.6.4.

2.4.3 Multi-component scalar field

Let $\phi(x) = (\phi_i(x))_{i=1,\ldots,n} \in V = \mathbb{R}^n$ be an n-component real scalar field. We denote the inner product on V by $(x,y) = \sum_{i=1}^{n} x_i y_i$, and the norm $|x|^2 = (x,x)$. Then, we consider the Lagrangian of the form,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - \frac{1}{2}m^2|\phi(x)|^2 - \frac{\lambda}{4!}\left(|\phi(x)|^2\right)^2 , \tag{2.4.10}$$

and the Euler–Lagrange equation gives rise to the n-component analog of the Klein–Gordon equation. This Lagrangian is invariant under the $\mathrm{O}(n)$ transformation:

$$\phi_i(x) \longrightarrow O_i{}^j \phi_j(x) , \qquad O \in \mathrm{O}(n) . \tag{2.4.11}$$

Therefore, we can discuss the Noether current for the multi-component real scalar.

Exercise 2.8. *Derive the Noether current for the n-component real scalar field theory associated with $\mathrm{O}(n)$ symmetry. We can similarly consider the n-component complex scalar field, $\phi(x) \in \mathbb{C}^n$ with the Lagrangian invariant under the $\mathrm{U}(n) = \mathrm{U}(1) \times \mathrm{SU}(n)$ transformation. Derive the Noether current for this case as well.*

2.4.4 Non-linear sigma model

The non-linear sigma model is a natural generalization of the previous example: One can consider the curved manifold as a target space V with the metric $g = (g_{ij})_{i,j=1,\ldots,\dim V}$,

$$\mathscr{L} = \frac{1}{2}g_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j = \frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) , \tag{2.4.12}$$

where we denote the inner product with the metric g_{ij} by $(\cdot,\cdot)$ in general. For the moment, we do not incorporate the potential term for simplicity. In the case of $\mathsf{d} = 2$, the mass dimension of the field is $[\phi] = 0$, so that one can consider arbitrary metric as a function of ϕ.

Polyakov action

In general, one can consider the curved spacetime M with the metric $h = (h_{\mu\nu})_{\mu,\nu=0,\ldots,D}$ and $\det h < 0$ (Lorentzian signature). In this case, the Lagrangian (2.4.12) is given by

$$\mathscr{L} = \frac{1}{2}h_{\mu\nu}g_{ij}(\phi)\partial^\mu \phi^i \partial^\nu \phi^j = \frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) , \tag{2.4.13}$$

and the corresponding action is called the Polyakov action (The convention is slightly different from the standard one). The field ϕ defines a map from the spacetime to the target space, $\phi : M \longrightarrow V$, which parametrizes the embedding of d-dimensional world volume within the target space V. In this context, the invariance of the action under the deformation of the field $\phi^i \longrightarrow \phi^i + \xi^i$ is interpreted as the diffeomorphism in the target manifold V. For example, in the case of $\mathsf{d} = 2$, it describes the embedding of the world sheet of string, and only in this case, the non-linear terms are relevant. See Sec. 1.3.3.

$\mathrm{O}(n)$ **non-linear sigma model**

We can describe the non-linear sigma model with the curved target space without using the curved metric g explicitly in some cases. Let $g_{ij} = \delta_{ij}$ be a flat metric of $\mathbb{R}^n$. Then, we consider the constraint

$$(\phi, \phi) = \sum_{i=1}^{n} \phi_i^2 = 1 \,. \tag{2.4.14}$$

Namely, ϕ is a unit vector in $\mathbb{R}^n$, which defines the $(n-1)$-dimensional sphere S^{n-1} having a positive constant curvature. Therefore, the Lagrangian (2.4.12) with the unit vector constraint is the non-linear sigma model with the target space $V = S^{n-1}$. Since this model is invariant under the $\mathrm{O}(n)$ transformation (2.4.11), it is called the $\mathrm{O}(n)$ non-linear sigma model. We remark that the $(n-1)$-sphere has a realization as the quotient, $S^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$.

2.5 Spinor field

2.5.1 Dirac spinor

Let us discuss the spinor field Lagrangian in particular for $\mathsf{d} = 4$. We first consider the kinetic terms for $\mathbf{2}$ and $\mathbf{2}^*$ spinors:

$$\mathrm{i}\eta^\dagger \sigma^\mu \partial_\mu \eta \,, \qquad \mathrm{i}\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi \,, \tag{2.5.1}$$

which are combined into the Dirac kinetic term,

$$\mathscr{L} = \mathrm{i}\bar{\psi}\gamma^\mu \partial_\mu \psi \,. \tag{2.5.2}$$

This can be written in more symmetric way through integration by parts:

$$\mathscr{L} = \frac{1}{2}\left(\mathrm{i}\bar{\psi}\gamma^\mu \partial_\mu \psi - \mathrm{i}\partial_\mu \bar{\psi}\gamma^\mu \psi\right) . \tag{2.5.3}$$

We here define the Dirac spinor and the gamma matrices,

$$\psi = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix} , \qquad \gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} . \tag{2.5.4}$$

The Dirac conjugate is defined as

$$\bar{\psi} = \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.5.5}$$

We remark the relation for the gamma matrices,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \,, \tag{2.5.6}$$

which is invariant under the Lorentz transformation, $\gamma^\mu \to \Lambda^\mu{}_\nu \gamma^\nu$, $\Lambda \in \mathrm{SO}(1,3)$. Hence, the choice of the gamma matrices is not unique. The expression shown in (2.5.4) is called the Weyl (chiral) basis.

We also define

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.5.7}$$

and the chiral projection operators,

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5.8}$$

Then, we obtain the left-handed and right-handed spinors,

$$\psi_{\mathrm{R}} = \frac{1+\gamma^5}{2}\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad \psi_{\mathrm{L}} = \frac{1-\gamma^5}{2}\psi = \begin{pmatrix} 0 \\ \eta^{\dot{\alpha}} \end{pmatrix}. \tag{2.5.9}$$

This decomposition is called the chiral decomposition, which can be discussed in even dimensions in general.

Symmetry

The kinetic term is invariant under the U(1) transformations,

$$\mathrm{U}(1)_{\mathrm{L}}: \ \eta \longrightarrow e^{\mathrm{i}\theta_{\mathrm{L}}}\eta, \qquad \mathrm{U}(1)_{\mathrm{R}}: \ \xi \longrightarrow e^{\mathrm{i}\theta_{\mathrm{R}}}\xi, \tag{2.5.10}$$

which are equivalent to

$$\mathrm{U}(1)_{\mathrm{V}}: \quad \psi \longrightarrow e^{\mathrm{i}(\theta_{\mathrm{L}}+\theta_{\mathrm{R}})/2}\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{-\mathrm{i}(\theta_{\mathrm{L}}+\theta_{\mathrm{R}})/2}, \tag{2.5.11a}$$

$$\mathrm{U}(1)_{\mathrm{A}}: \quad \psi \longrightarrow e^{\mathrm{i}\gamma^5(-\theta_{\mathrm{L}}+\theta_{\mathrm{R}})/2}\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{\mathrm{i}\gamma^5(-\theta_{\mathrm{L}}+\theta_{\mathrm{R}})/2}. \tag{2.5.11b}$$

Therefore, the kinetic term of the Dirac spinor (2.5.2) has $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$ symmetry, called **chiral symmetry**. If one consider n sets of the Dirac spinor (n flavor system), it consequently has $\mathrm{U}(n)_{\mathrm{V}} \times \mathrm{U}(n)_{\mathrm{A}}$ symmetry.

Exercise 2.9 (Vector and axial symmetries)**.** *Derive the Noether currents associated with $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ symmetry from the Lagrangian* (2.5.2)*:*

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi, \tag{2.5.12}$$

which are called the vector and axial-vector currents. The corresponding conserved charges are then given by

$$Q = \begin{cases} \displaystyle\int d^D x\, \psi^\dagger\psi = \int d^D x \left(\xi^\dagger\xi + \eta^\dagger\eta\right) & (\mathrm{U}(1)_V) \\ \displaystyle\int d^D x\, \psi^\dagger\gamma^5\psi = \int d^D x \left(\xi^\dagger\xi - \eta^\dagger\eta\right) & (\mathrm{U}(1)_A) \end{cases} \tag{2.5.13}$$

Mass term

We can consider the mass term for the Dirac spinor

$$-m\bar{\psi}\psi = -m(\xi^\dagger\eta + \eta^\dagger\xi), \tag{2.5.14}$$

which is invariant only under $\mathrm{U}(1)_{\mathrm{V}}$; This mass term violates the $\mathrm{U}(1)_{\mathrm{A}}$ symmetry. Then, combining with the kinetic term (2.5.2), the total Lagrangian is given by

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi. \tag{2.5.15}$$

Exercise 2.10 (Dirac and Weyl equations)**.** *Derive the Euler–Lagrange equation,*

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0\,, \tag{2.5.16}$$

which is known as ***Dirac equation****. Then, if* $m = 0$*, confirm that the Dirac equation splits into* ***Weyl equations*** *for* ξ *and* η*,*

$$\mathrm{i}\sigma^\mu \partial_\mu \eta = 0\,, \qquad \mathrm{i}\bar{\sigma}^\mu \partial_\mu \xi = 0\,. \tag{2.5.17}$$

2.5.2 Interaction term

Let us consider the interaction term for the Dirac spinor. One of the most important examples is the **Nambu–Jona-Lasinio (NJL) model**:

$$\mathscr{L}_{\mathrm{NJL}} = \bar{\psi}\mathrm{i}\gamma^\mu \partial_\mu \psi + G\Big(\big(\bar{\psi}\psi\big)^2 + \big(\bar{\psi}\mathrm{i}\gamma^5\psi\big)^2\Big)\,, \tag{2.5.18}$$

where G is the coupling constant. This kind of the interaction term is called the four-fermi interaction. In fact, the interaction term of the NJL model is invariant under $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$ transformation (2.5.11), so that the NJL model has the chiral symmetry.[9]

Exercise 2.11 (Chiral symmetry of the NJL model)**.**

1. *Show* $\mathrm{e}^{\mathrm{i}\theta\gamma^5} = \cos\theta + \mathrm{i}\gamma^5 \sin\theta$*. One may use the formulas:*

$$\sin x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,, \qquad \cos x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\,, \tag{2.5.19}$$

   together with $\big(\gamma^5\big)^2 = 1$*.*

2. *Show that the interaction part of the NJL model is invariant under* $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ *transformation* (2.5.11)*.*

In addition to the four-fermi interaction, there is also an interaction between the fermion (spinor) and the scalar field, called the **Yukawa interaction**, $\bar{\psi}\psi\phi$. In this coupling, the scalar field plays a role of the mass parameter for the spinor (originally introduced as a meson). See Sec. 3.5.2 and Sec. 6.2.3.

2.5.3 Majorana spinor

We consider the charge conjugation operation:

$$\mathrm{C}:\ \psi = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix} \longrightarrow \psi^{\mathrm{C}} = \begin{pmatrix} \eta^*_\alpha \\ \xi^{*\dot{\alpha}} \end{pmatrix} = \mathrm{i}\gamma^2\psi^* =: \mathrm{C}\bar{\psi}^{\mathrm{T}} \tag{2.5.20}$$

where the charge conjugation matrix is given by

$$\mathrm{C} = \mathrm{i}\gamma^2\gamma^0 \quad \Longrightarrow \quad \mathrm{C}^2 = -1\,. \tag{2.5.21}$$

[9]This is true only at the classical level. Implementing the quantum correction, it is known that the chiral symmetry of the NJL model would be spontaneously broken. This phenomenon (the realized vacuum state violates the symmetry of the original Lagrangian) is called the **spontaneous symmetry breaking (SSB)** of the chiral symmetry. See Chapter 6 for details.

Exercise 2.12 (Charge conjugation)**.** *Show the gamma matrices transforms under the charge conjugation:*

$$\mathrm{C}^{-1}\gamma^{\mu}\mathrm{C} = -\gamma^{\mu T} . \tag{2.5.22}$$

Then, we consider the **Majorana spinor**,[10]

$$\psi_{\mathrm{M}} = \begin{pmatrix} \eta^*_{\alpha} \\ \eta^{\dot{\alpha}} \end{pmatrix} \tag{2.5.23}$$

which is invariant under the charge conjugation, $\psi_{\mathrm{M}} = \psi^{\mathrm{C}}_{\mathrm{M}}$. The Lagrangian is given by

$$\mathscr{L} = \frac{1}{2}\bar{\psi}_{\mathrm{M}} \mathrm{i}\gamma^{\mu}\partial_{\mu}\psi_{\mathrm{M}} - \frac{1}{2}m\bar{\psi}_{\mathrm{M}}\psi_{\mathrm{M}} = \eta^{\dagger}\mathrm{i}\sigma^{\mu}\partial_{\mu}\eta - \frac{m}{2}\left(\eta^{\mathrm{T}}\epsilon\eta - \eta^{\dagger}\epsilon\eta^*\right) . \tag{2.5.24}$$

The mass term is explicitly written as

$$\eta^{\mathrm{T}}\epsilon\eta = \eta^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\eta^{\dot{\beta}} = \eta^1\eta^2 - \eta^2\eta^1 , \tag{2.5.25}$$

which implies that we should consider the spinor field as an anti-commuting variable, called **Grassmann number**, $\eta^1\eta^2 = -\eta^2\eta^1$. Otherwise, one cannot consider the mass term for the Majorana spinor.[11] The complex conjugate is now defined as $(\eta\xi)^* = \xi^*\eta^*$.

We remark that the Majorana mass term violates the $\mathrm{U}(1)_{\mathrm{L}}$ symmetry described in (2.5.10) into $\mathbb{Z}_2 = \mathrm{O}(1)$: $\eta \to -\eta$.[12] This means that the particle number is not a conserved quantity in the presence of the Majorana mass term. The particle number parity $(-1)^N$ is still well-defined, but it is not a Noether charge, since $\mathbb{Z}_2 = \mathrm{O}(1)$ is a discrete symmetry.

2.5.4 Grassmannian calculus

Let us discuss the calculus of the Grassmann numbers,

$$\{\theta_i, \theta_j\} = 0 , \qquad i, j = 1, \ldots, n . \tag{2.5.26}$$

From this property, it turns out that Grassmann numbers are nilpotent,

$$\theta_i^2 = 0 . \tag{2.5.27}$$

Derivative and integral

We define the derivative in the same way as the ordinary commutative c-number,

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta_{i,j} . \tag{2.5.28}$$

We remark the relation $[\partial_{z_i}, z_j] = \delta_{i,j}$ for the ordinary number is replaced with the anticommutation relation,

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{i,j} . \tag{2.5.29}$$

[10]This is also known as **Nambu spinor** in the context of Bogoliubov–de Gennes (BdG) Hamiltonian describing excitation in the superconductor.

[11]Recall the spin-statistics theorem in Sec. 2.1.1.

[12]If one construct the Majorana spinor from the spinor ξ, the corresponding mass term similarly violates the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry.

Then, the integral with the Grassmann number is defined as follows:

$$\int \mathrm{d}\theta_i \, \theta_j = \delta_{i,j} \,, \qquad \int \mathrm{d}\theta_i \, 1 = 0 \,. \tag{2.5.30}$$

Namely, the derivative and the integral are essentially the same for Grassmann numbers.

Exercise 2.13 (Jacobian for the Grassmann variables)**.** *Show that, under the transformation* $U : \theta_i \mapsto U_{ij}\theta_j$*, the integral measure behaves as*

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \;\longmapsto\; \int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, (\det U)^{-1} \,, \tag{2.5.31}$$

and compare with the Jacobian of the ordinary commutative variables,

$$U : z_i \mapsto U_{ij} z_j \,, \qquad \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \;\longmapsto\; \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \, (\det U)^{+1} \,. \tag{2.5.32}$$

2.6 Local symmetry and gauge field

2.6.1 Gauge transformation

We have considered the U(1) transformation (2.4.6), which does not depend on the spacetime coordinate $x \in M$ (**global transformation**). Let us similarly consider the spacetime dependent transformation (**local transformation**) as follows:

$$\phi(x) \;\longrightarrow\; \mathrm{e}^{\mathrm{i}\theta(x)} \phi(x) \,. \tag{2.6.1}$$

This is called **U(1) gauge transformation**. Under such a local transformation, the derivative term is not invariant,

$$\partial_\mu \phi(x) \;\longrightarrow\; \partial_\mu \left(\mathrm{e}^{\mathrm{i}\theta(x)} \phi(x) \right) = \mathrm{e}^{\mathrm{i}\theta(x)} \partial_\mu \phi(x) + \mathrm{i}(\partial_\mu \theta(x)) \mathrm{e}^{\mathrm{i}\theta(x)} \phi(x) \,. \tag{2.6.2}$$

In order to compensate the extra factor, we then define **covariant derivative**:

$$D_\mu \phi(x) = (\partial_\mu - \mathrm{i}eA_\mu(x))\phi(x) \,, \tag{2.6.3}$$

where the gauge transformation of the vector field $A_\mu(x)$, called **gauge field**, is given as follows:

$$A_\mu(x) \;\longrightarrow\; A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x) \,. \tag{2.6.4}$$

Exercise 2.14 (U(1) gauge symmetry)**.**

1. *Show that the covariant derivative term behaves under the U(1) transformation as*

$$D_\mu \phi(x) \;\longrightarrow\; \mathrm{e}^{\mathrm{i}\theta(x)} D_\mu \phi(x) \,. \tag{2.6.5}$$

2. *Confirm that the Lagrangian*

$$\mathscr{L} = (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \tag{2.6.6}$$

is invariant under the local U(1) transformation.

3. *Apply the same argument to the Dirac field, and show that the Lagrangian*

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi \tag{2.6.7}$$

is invariant under the local U(1) transformation.

2.6.2 Maxwell term

We consider the kinetic term for the gauge field $A_\mu(x)$. The gauge invariant combination of the first derivative terms is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{2.6.8}$$

which is called the **field strength**. Identifying the gauge field with the scalar and vector potentials in $\mathsf{d} = 4$, $A = (\phi, A_{i=1,2,3})$, each component gives the electric and magnetic field,

$$F_{0i} = \partial_0 A_i - \partial_i \phi = E_i \,, \qquad F_{ij} = \partial_i A_j - \partial_j A_i = -\epsilon_{ijk} B^k \,, \tag{2.6.9}$$

with

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{2.6.10}$$

Exercise 2.15 (Field strength)**.** *Show that the field strength is obtained from the commutation relation of the covariant derivative as follows:*

$$\frac{1}{-\mathrm{i}e}[D_\mu, D_\nu]\phi(x) = F_{\mu\nu}\phi(x) \,. \tag{2.6.11}$$

In order to discuss the Lagrangian, we should contract the spacetime indices to construct the Lorentz scalar. Then, the gauge invariant kinetic term is given as follows:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right) , \tag{2.6.12}$$

which is called the **Maxwell Lagrangian**. In addition to the kinetic term, we also introduce the source term,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \,, \tag{2.6.13}$$

where we denote the current by $J^\mu = (\rho, J^{i=1,2,3})$ obeying the conservation law $\partial_\mu J^\mu = 0$. We remark that the source term behaves under the gauge transformation as

$$A_\mu J^\mu \longrightarrow \left(A_\mu + \frac{1}{e}\partial_\mu\theta\right) J^\mu = A_\mu J^\mu + \frac{1}{e}\partial_\mu(\theta J^\mu) - \frac{1}{e}\theta\partial_\mu J^\mu \,. \tag{2.6.14}$$

Exercise 2.16 (Maxwell equations)**.** *Derive the Euler–Lagrangian equation from the Lagrangian* (2.6.13)*:*

$$\partial_\mu F^{\mu\nu} = J^\nu \,. \tag{2.6.15}$$

Then, together with the Bianchi identity,

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \,, \tag{2.6.16}$$

compare with the Maxwell equations,

$$\vec{\nabla}\cdot\vec{E} = \rho \,, \qquad \vec{\nabla}\times\vec{B} - \partial_0\vec{E} = \vec{J} \,, \tag{2.6.17a}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \,, \qquad \partial_0\vec{B} + \vec{\nabla}\times\vec{E} = 0 \,. \tag{2.6.17b}$$

2.6.3 Geometry of gauge theory

We discuss a geometric formulation of gauge theory. The gauge field and the field strength are written as the differential forms:

$$A = A_\mu dx^\mu \,, \qquad F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \,, \tag{2.6.18}$$

where we denote the exterior derivative by d with the $\wedge$-product, s.t., $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. The p-form differential is in general written as

$$\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Omega^p(M) \,, \tag{2.6.19}$$

where we denote a set of p-forms on M by $\Omega^p(M)$. From this point of view, the gauge field plays a role of the connection on the spacetime M, and the field strength is the corresponding curvature.

The Maxwell Lagrangian is then written using the differential form as follows:

$$\mathscr{L} = -\frac{1}{2} F \wedge \star F - A \wedge \star J \,, \tag{2.6.20}$$

where the Hodge dual operator is a map, $\star : \Omega^p(M) \longrightarrow \Omega^{\mathsf{d}-p}(M)$. For the p-form differential (2.6.19), it is given by

$$\star\alpha = \frac{1}{(\mathsf{d}-p)!} (\star\alpha)_{i_{p+1} \cdots i_{\mathsf{d}}} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_{\mathsf{d}}} \,, \qquad (\star\alpha)_{i_{p+1} \cdots i_{\mathsf{d}}} = \frac{1}{p!} \epsilon_{i_1 \cdots i_{\mathsf{d}}} \alpha^{i_1 \cdots i_p} \,. \tag{2.6.21}$$

In this formalism, the current conservation is described as

$$d \star J = 0 \,. \tag{2.6.22}$$

Then, the Euler–Lagrange equation and the Bianchi identity are given as

$$d \star F = \star J \,, \qquad dF = 0 \,. \tag{2.6.23}$$

We remark that the Bianchi identity immediately follows from the nilpotent property of the exterior derivative $d^2 = 0$. In particular, in the case of $\mathsf{d} = 4$, the Hodge dual maps $\Omega^2(M) \longrightarrow \Omega^2(M)$. Therefore, we obtain the dual curvature

$$\star : \; F_{\mu\nu} \longmapsto \star F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \,. \tag{2.6.24}$$

Exercise 2.17 (Topological term)**.**

1. *Show the component of the dual curvature is given by*

$$\star F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} \,, \tag{2.6.25}$$

   and compare with the original expression (2.6.10).

2. *Show that another gauge invariant Lagrangian in* $\mathsf{d} = 4$ *takes a form,*

$$\mathscr{L} = -\frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \frac{1}{2} \vec{E} \cdot \vec{B} \,. \tag{2.6.26}$$

3. *Consider the Lagrangian (2.6.26) written in the differential form expression. Then show that it is written in a total derivative form, and confirm that it does not contribute to the Euler–Lagrange equation,*

$$\mathscr{L} = -\frac{1}{2} F \wedge F = -\frac{1}{2} d(AdA) \,. \tag{2.6.27}$$

In this sense, this term is called the topological term, which is invariant under the continuous deformation of the field. This is related to the fact that the n-th ***Chern class*** *is written as an exterior derivative of the* ***Chern–Simons*** *$(2n-1)$**-form**.*

Selfdual/anti-selfdual tensor

The curvature splits into the selfdual and the anti-selfdual tensors in $\mathsf{d} = 4$,[13]

$$F = F_+ + F_- \,, \qquad F_\pm := \frac{1}{2}(F \pm \star F) = \pm \star F \,. \tag{2.6.28}$$

In fact, this corresponds to the $(1,0)$ and $(0,1)$ representations of the Lorentz group. See the Lorentz group representations (2.1.20).

We then consider the (anti-)selfdual equation in $\mathsf{d} = 4$,

$$\star F = \pm F \qquad \Longleftrightarrow \qquad F_\mp := \frac{1}{2}(F \mp \star F) = 0 \,. \tag{2.6.29}$$

In fact, the (anti-)selfdual tensor turns out to be a solution to the Euler–Langange equation in the absense of the source term,

$$d \star F \overset{\text{(A)SD}}{=} \pm dF \overset{\text{Bianchi}}{=} 0 \,. \tag{2.6.30}$$

2.6.4 Non-Abelian gauge theory

One may consider the local version of the higher-rank transformation of the field discussed in Sec. 2.4.3. For example, let $V = \mathbb{C}^n$, and consider $\mathrm{U}(n) = \mathrm{U}(1) \times \mathrm{SU}(n)$ local transformation. Since the $\mathrm{U}(1)$ part has been already discussed above, we may focus on the $\mathrm{SU}(n)$ part.

From now on, we consider the gauge transformation with generic simple Lie group G,

$$\phi(x) \longrightarrow U(x)\phi(x) \,, \qquad A_\mu \longrightarrow U A_\mu U^{-1} - \frac{1}{\mathrm{i}g} U \partial_\mu U^{-1} \,, \qquad U(x) \in G \,, \tag{2.6.31}$$

where g is the coupling constant. The Lie group element is parametrized as follows:

$$U(x) = \exp\left(\mathrm{i}\theta^a(x) t^a\right) , \tag{2.6.32}$$

where $(t^a)_{a=1,\ldots,\dim G}$ are the generators of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ with the commutation relation[14]

$$\left[t^a, t^b\right] = \mathrm{i} f^{ab}{}_c t^c \,, \tag{2.6.33}$$

[13] For the moment, we consider the Euclidian signature. For the Lorentzian signature, we should put the imaginary unit $\mathrm{i} = \sqrt{-1}$ for the definition. See also (2.1.30).

[14] We apply the convention, s.t., the Lie algebra generators are hermitian. One often uses the anti-hermitian generators in the literature, in which the imaginary unit $\mathrm{i} = \sqrt{-1}$ does not explicitly appear: The commutation relation is written as $\left[t^a, t^b\right] = f^{ab}{}_c t^c$, and the Lie group element is parametrized as $g(x) = \exp\left(\theta^a(x) t^a\right)$.

and with the normalization

$$\operatorname{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} . \tag{2.6.34}$$

The derivative term $U^{-1}\partial U$ is called the **Maurer–Cartan one-form**, which takes a value in the Lie algebra $\mathfrak{g}$. Therefore, the gauge field is expanded with the Lie algebra generators,

$$A_\mu = \sum_{a=1}^{\dim G} A_\mu^a t^a . \tag{2.6.35}$$

In the case of $G = \mathrm{SU}(2)$, the generators are given by (a half of) the Pauli matrices, $t^a = \sigma^a/2$ for $a = 1, 2, 3$. We remark that the mass term for the gauge field $\frac{m^2}{2} \operatorname{tr} A_\mu A^\mu$ breaks the gauge symmetry, as is not compatible with the gauge transformation (2.6.31).

Exercise 2.18 (Non-Abelian gauge symmetry)**.**

1. *Show that the covariant derivative term (in the fundamental representation of G) transforms under the G-gauge transformation as follows:*

$$D_\mu \phi(x) := (\partial_\mu - igA_\mu(x))\phi(x) \xrightarrow{(2.6.31)} U(x)D_\mu\phi(x) . \tag{2.6.36}$$

2. *Assuming $\theta^a \ll 1$, the group element* (2.6.32) *has an expansion*

$$U(x) = 1 + \mathrm{i}\theta^a(x)t^a + O(\theta^2) . \tag{2.6.37}$$

Then, show that the infinitesimal version of the the gauge transformation (2.6.31) *is given by*

$$\phi(x) \longrightarrow \phi(x) + \mathrm{i}\theta^a(x)t^a\phi(x) , \tag{2.6.38a}$$

$$A_\mu^a \longrightarrow A_\mu^a + \frac{1}{g}\partial_\mu\theta^a - f_{bc}{}^a A^b\theta^c =: A_\mu^a + \frac{1}{g}D_\mu\theta^a , \tag{2.6.38b}$$

where D_μ here is the covariant derivative acting on a field in the adjoint representation of G. In the matrix form, $\theta = \theta^a t^a$, we insted have

$$A_\mu \longrightarrow A_\mu + \frac{1}{g}\partial^\mu\theta + \mathrm{i}[A_\mu, \theta] = A_\mu + \frac{1}{g}D_\mu\theta . \tag{2.6.39}$$

3. *Show that the field strength (curvature) constructed from the covariant derivative (See* (2.6.11)*) takes a form,*

$$F_{\mu\nu} = \frac{1}{-\mathrm{i}g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] . \tag{2.6.40}$$

4. *Show that the field strength behaves under the G-gauge transformation as follows:*

$$F_{\mu\nu} \xrightarrow{(2.6.31)} UF_{\mu\nu}U^{-1} . \tag{2.6.41}$$

We can construct the gauge invariant Lagrangian similarly to the U(1) theory (2.6.13),

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} , \tag{2.6.42}$$

which is called **Yang–Mills (YM) Lagrangian**. In fact, this Lagrangian contains non-linear terms of the gauge field A_μ, which describe the self-interaction of the gauge field. See Sec. 5.2.5.

Differential form description

We can formulate the G-gauge theory with the differential form description similarly to Sec. 2.6.3. In general, the gauge field is described as a $\mathfrak{g}$-valued one-form on M,

$$A \in \Omega^1(M, \mathfrak{g}) \,, \tag{2.6.43}$$

with G-transformation

$$G: \; A \longrightarrow UAU^{-1} - \frac{1}{\mathrm{i}g} U dU^{-1} \,, \qquad U \in G \,. \tag{2.6.44}$$

Then, the curvature is given by a $\mathfrak{g}$-valued two-form,

$$F = DA = (d - \mathrm{i}gA\wedge)A = dA - \mathrm{i}gA \wedge A \in \Omega^2(M, \mathfrak{g}) \,, \tag{2.6.45}$$

with the G-transformation,

$$G: \; F \longrightarrow UFU^{-1} \,. \tag{2.6.46}$$

Namely, A (more precisely, the covariant derivative D) and F transform in the adjoint representation of G. This is mathematically established as the **principal bundle** with the structure group G (principal G-bundle).

The Lagrangian is given as before

$$\mathscr{L}_{\mathrm{YM}} = - \operatorname{tr} (F \wedge \star F) \,, \tag{2.6.47}$$

with the Euler–Lagrange equation,

$$D \star F = 0 \,. \tag{2.6.48}$$

This is a non-linear second order PDE, so that it is difficult to solve it in general. We remark that, in $\mathsf{d} = 4$, the (anti-)selfdual curvature

$$\star F = \pm F \tag{2.6.49}$$

solves this equation due to the (generalized version of) Bianchi identity,

$$D \star F \overset{\text{(A)SD}}{=} \pm DF \overset{\text{Bianchi}}{=} 0 \,. \tag{2.6.50}$$

The (anti-)selfdual equation is still non-linear, but is a first order PDE, which is easier to deal with. The equation (2.6.49) is called the **(anti-)self-dual Yang–Mills ((A)SDYM) equation**, and its solution is called the **instanton**, which would play an important role to understand the non-perturbative aspects of four-dimensional quantum gauge theory.

Topological term

In the case of $\mathsf{d} = 4$, there is another gauge invariant Lagrangian in addition to the YM Lagrangian,

$$\mathscr{L} = \operatorname{tr} F \wedge F = d \operatorname{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) =: d\mathrm{CS}_3 \,, \tag{2.6.51}$$

where CS_3 is called the Chern–Simons three-form. Since this term does not contribute to the Euler–Lagrange equation, it is called the topological term as before.

2.7 Gauge theory description of curved manifold

We start with generic metric $g = (g_{\mu\nu})_{\mu,\nu=0,\ldots,D}$. Since it is a symmetryc matrix depending on a spacetime coordinate $x \in M$, one can diagonalize it using the local $\mathrm{SO}(1,D)$ transformation,

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{2.7.1}$$

where $\eta = (\eta_{ab})_{a,b=0,\ldots,D}$ is the Lorentzian metric (2.1.1), and $e = (e_\mu{}^a)_{\mu,0=1,\ldots,D} \in \mathrm{SO}(1,D)$ is called the **vielbein**. Using the vielbein, one can describe the curved manifold with locally flat patches. For the moment, we apply the Greek indices $\mu, \nu, \ldots$ as the curved manifold indices, and the Latin indices $a, b, \ldots$ to the locally flat ones.

We then consider the local Lorentz transformation. For this purpose, we recall that the Dirac spinor behaves under the Lorentz transformation,

$$\psi \longrightarrow \exp\left(-\frac{\mathrm{i}}{4}\epsilon^{ab}(x)\gamma_{ab}\right)\psi \tag{2.7.2}$$

where we define

$$\gamma_{ab} = \frac{\mathrm{i}}{2}[\gamma_a, \gamma_b] \,. \tag{2.7.3}$$

Namely, $(\gamma_{ab})_{a,b=0,\ldots,D}$ are the generators of the Lorentz group in the spinor representation. Applying the same argument to Sec. 2.6.4, we introduce the spin connection to define the covariant derivative,

$$\nabla_a = e^\mu{}_a(\partial_\mu + \mathrm{i}\omega_\mu) \,, \qquad \omega_\mu = \frac{1}{2}\omega_\mu^{ab}\gamma_{ab} \,. \tag{2.7.4}$$

This spin connection actually plays a role of the gauge field for the local Lorentz gauge symmetry. It is in fact related to the Christoffel symbol as

$$\omega_\mu{}^a{}_b = e_\nu{}^a \Gamma^\nu_{\mu\lambda} e^\lambda{}_b - e_\nu{}^a \partial_\mu e^\nu{}_b \,, \tag{2.7.5}$$

which is essentially the gauge transformation as shown in (2.6.31).

This formalism plays an essential role to consider the spinor field on the curved manifold. In general, the gamma matrices obey the generalized version of the relation (2.5.6) as

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \,. \tag{2.7.6}$$

Using the vielbein, one can convert the gamma matrices to those for the flat space, $\gamma_\mu = e_\mu{}^a \gamma_a$ with $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. Therefore, the Lagrangian for the spinor on the curved manifold is given by

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^a \nabla_a - m)\psi \,. \tag{2.7.7}$$

Differential form description

We apply the differential form formalism to the local Lorentz gauge field. We denote the vielbein and the spin connection one-forms by

$$e^a = e_\mu{}^a dx^\mu \,, \qquad \omega^{ab} = \omega_\mu^{ab} dx^\mu \,. \tag{2.7.8}$$

Applying the covariant derivative, we obtain the **torsion** and the **Riemann curvature** two-forms as follows:

$$T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{\mu\nu}\, dx^\mu \wedge dx^\nu \,, \tag{2.7.9a}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu \,, \tag{2.7.9b}$$

which are converted to the standard convention,

$$T^\rho{}_{\mu\nu} = e^\rho{}_a T^a{}_{\mu\nu} \,, \qquad R^\rho{}_{\sigma\mu\nu} = e^\rho{}_a e_\sigma{}^b R^a{}_{b\mu\nu} \,. \tag{2.7.10}$$

The **Ricci curvature** is constructed from the Riemann curvature,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = e^\rho{}_a e_\mu{}^b R^a{}_{b\rho\nu} \,. \tag{2.7.11}$$

In $\mathsf{d} = 4$, we may consider the following topological terms:

$$e^a \wedge e^b \wedge R_{ab} \,, \qquad T^a \wedge T_a \,, \qquad R^a{}_b \wedge R^b{}_a \,, \qquad \epsilon_{abcd} R^{ab} \wedge R^{cd} \tag{2.7.12}$$

The third one is the **Pontryagin class**, and the four one is the **Euler class** of the tangent bundle. See, for example, [Nak03, Sec. 11.4] for details. The linear combination of the first and the second ones, $T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab}$ is called the **Nieh–Yan four-form**. The corresponding (gravitational) Chern–Simons three-forms are given as

$$R^a{}_b \wedge R^b{}_a = d\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3} \omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a \right) , \tag{2.7.13a}$$

$$T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab} = d(e^a \wedge T_a) = d\left(e^a \wedge de_a + e^a \wedge \omega^b{}_a \wedge e_b \right) . \tag{2.7.13b}$$

Chapter 3

Path integral quantization

In this Chapter, we discuss the path integral formalism to quantize the field theories based on the Lagrangian formalism discussed in Chapter 2. We also discuss how to deal with the interacting field theory based on the perturbation expansion, and introduce the Feynman rule to efficiently describe the expansion.

3.1 Generating functional

The path integral is formulated as an amplitude from the initial state at t_i to the final state at t_f. We in particular interested in the situation $(t_\mathrm{i}, t_\mathrm{f}) \to (-\infty, +\infty)$, s.t., the initial and final states are given by the vacuum (ground state) $|0\rangle$.[1] Then, the corresponding amplitude in the path integral formalism (also called the **partition function** in the analogy with the statistical mechanics) is given by

$$Z := \langle 0|0\rangle = \int D\phi \, \exp\left(\mathrm{i} \int \mathrm{d}^\mathsf{d} x \, \mathscr{L}(\phi, \partial\phi)\right), \tag{3.1.1}$$

where the path integral measure is a formal product of the measures at each spacetime point,

$$D\phi = \prod_{x \in M} \mathrm{d}\phi(x) \,. \tag{3.1.2}$$

With this formalism, we consider the n-point correlation function,

$$\begin{aligned} \langle \mathrm{T}[\phi(x_1) \cdots \phi(x_n)] \rangle &= \frac{\langle 0| \, \mathrm{T}[\phi(x_1) \cdots \phi(x_n)] \, |0\rangle}{\langle 0|0\rangle} \\ &= \frac{1}{Z} \int D\phi \, \phi(x_1) \cdots \phi(x_n) \, \exp\left(\mathrm{i} \int \mathrm{d}^\mathsf{d} x \, \mathscr{L}(\phi, \partial\phi)\right), \end{aligned} \tag{3.1.3}$$

where we apply the **time-ordering product**,

$$\mathrm{T}[\phi(x_1) \cdots \phi(x_n)] = \phi(x_{i_n}) \cdots \phi(x_{i_1}) \quad \text{s.t.,} \quad \begin{gathered} (i_k)_{k=1,\ldots,n} = \{1, \ldots, n\} \,, \\ +\infty > t_{i_n} > \cdots > t_{i_1} > -\infty \,. \end{gathered} \tag{3.1.4}$$

In order to compute the correlation function, we consider the **generating functional**,

$$Z[J] = \int D\phi \, \exp\left(\mathrm{i} \int \mathrm{d}^\mathsf{d} x \, (\mathscr{L}(\phi, \partial\phi) + J(x)\phi(x))\right), \tag{3.1.5}$$

[1] One can justify this argument through the Euclidianization: Applying the imaginary time formalism $t \to -\mathrm{i}\tau$, the phase factor $\mathrm{e}^{\mathrm{i}E(t_\mathrm{i} - t_\mathrm{f})} \to \mathrm{e}^{E(\tau_\mathrm{i} - \tau_\mathrm{f})} \to 0$ if $E \neq 0$ at $(\tau_\mathrm{i}, \tau_\mathrm{f}) \to (-\infty, +\infty)$.

where $J(x)$ is the external source field. In this convention, the partition function is given by $Z = Z[J = 0]$. Then, the functional derivative with the source field is given by

$$\frac{\delta Z[J]}{\delta J(x)} = \mathrm{i} \int D\phi \, \phi(x) \, \exp\left(\mathrm{i} \int \mathrm{d}^{\mathsf{d}} x \left(\mathscr{L}(\phi, \partial\phi) + J(x)\phi(x) \right) \right) , \tag{3.1.6}$$

so that we obtain the n-point function via the functional derivatives,

$$\langle \mathrm{T}[\phi(x_1) \cdots \phi(x_n)] \rangle = \frac{\mathrm{i}^{-n}}{Z[J]} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J \to 0} . \tag{3.1.7}$$

In many cases, we are interested in the **connected part** (also known as the **cumulant**) of the correlation function. For example, the connected part of the lower order correlation functions is given by

$$\langle \phi(x_1)\phi(x_2) \rangle_{\mathrm{c}} = \langle \phi(x_1)\phi(x_2) \rangle - \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle \,, \tag{3.1.8a}$$

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle_{\mathrm{c}} = & \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle \\ & - \langle \phi(x_1) \rangle \langle \phi(x_2)\phi(x_3) \rangle - \langle \phi(x_2) \rangle \langle \phi(x_1)\phi(x_3) \rangle - \langle \phi(x_3) \rangle \langle \phi(x_1)\phi(x_2) \rangle \\ & + 2 \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle \langle \phi(x_3) \rangle \,, \end{aligned} \tag{3.1.8b}$$

where we do not show the time-ordering symbol for simplicity. We remark that, for the ϕ^4-theory, the odd-point correlation functions become zero due to the symmetry $\phi \to -\phi$ (see Exercise 3.3). Hence, for example, we have $\langle \phi(x_1)\phi(x_2) \rangle_{\mathrm{c}} = \langle \phi(x_1)\phi(x_2) \rangle$.

Exercise 3.1 (Generating functional of the connected correlation functions)**.** *Show that the connected part is obtained as*

$$\langle \mathrm{T}[\phi(x_1) \cdots \phi(x_n)] \rangle_c = \mathrm{i}^{-n} \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J \to 0} , \tag{3.1.9}$$

where we define the generating functional of the connected correlation function as follows:

$$W[J] = \log Z[J] \qquad \Longleftrightarrow \qquad Z[J] = \exp\left(W[J] \right) . \tag{3.1.10}$$

3.2 Free scalar field

Let us demonstrate the generating functional formalism with the free scalar theory (cf. (2.4.1)),

$$\mathscr{L}_0 = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 . \tag{3.2.1}$$

In this case, the generating functional (3.1.5) is given by

$$\begin{aligned} Z_0[J] &= \int D\phi \, \exp\left(\mathrm{i} \int \mathrm{d}^{\mathsf{d}} x \left(\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + J(x)\phi(x) \right) \right) \\ &= \int D\phi \, \exp\left(\mathrm{i} \int \mathrm{d}^{\mathsf{d}} x \left(-\frac{1}{2} \phi(x) \left(\partial_\mu \partial^\mu + m^2 \right) \phi(x) + J(x)\phi(x) \right) \right) . \end{aligned} \tag{3.2.2}$$

In order to evaluate this integral, let us introduce the Gaussian integral formulas as follows.

Exercise 3.2 (Gaussian integrals)**.**

1. *Show the following integral formulas (Gaussian integrals),*

$$\int_{\mathbb{R}} \mathrm{d}x \exp\left(-\frac{1}{2} x^2 \right) = \sqrt{2\pi} \,, \qquad \int_{\mathbb{C}} \mathrm{d}z \, \mathrm{d}z^* \exp\left(-|z|^2 \right) = 2\pi \,. \tag{3.2.3}$$

2. *Let A and B be positive semi-definite real symmetric and complex Hermitian matrices. Show the multi-dimensional analogs of the Gaussian integral,*

$$\int \prod_{i=1}^{n} \mathrm{d}x_i \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\,, \tag{3.2.4a}$$

$$\int \prod_{i=1}^{n} \mathrm{d}z_i\, \mathrm{d}z_i^* \exp\left(-\sum_{i,j=1}^{n} z_i^* B_{ij} z_j\right) = \frac{(2\pi)^n}{\det B}\,. \tag{3.2.4b}$$

Hint: These integrals are invariant under

$$x_i \longrightarrow O_{ij}x_j\,, \qquad A_{ij} \longrightarrow O_{ik}A_{kl}O^T_{lj}\,, \tag{3.2.5a}$$

$$z_i \longrightarrow U_{ij}z_j\,, \qquad B_{ij} \longrightarrow U_{ik}B_{kl}U^\dagger_{lj}\,, \tag{3.2.5b}$$

for $O \in \mathrm{O}(n)$ and $U \in \mathrm{U}(n)$, so that it is convenient to consider the basis diagonalizing A and B.

3. *Show the formulas*

$$\int \prod_{i=1}^{n} \mathrm{d}x_i \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} c_i x_i\right) = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} c_i A^{-1}_{ij} c_j\right), \tag{3.2.6a}$$

$$\int \prod_{i=1}^{n} \mathrm{d}z_i\, \mathrm{d}z_i^* \exp\left(-\sum_{i,j=1}^{n} z_i^* B_{ij} z_j + \sum_{i=1}^{n}(d_i^* z_i + z_i^* d_i)\right) = \frac{(2\pi)^n}{\det B} \exp\left(\sum_{i,j=1}^{n} d_i^* B^{-1}_{ij} d_j\right). \tag{3.2.6b}$$

We apply the Gaussian integral formulas to the path integral. We first rewrite the quadratic term in the path integral (3.2.2) as follows:

$$\begin{aligned}\int \mathrm{d}^{\mathsf{d}}x\, \phi(x)\big(\partial_\mu\partial^\mu + m^2\big)\phi(x) &= \int \mathrm{d}^{\mathsf{d}}x\, \mathrm{d}^{\mathsf{d}}y\, \phi(x)\big(\partial_\mu\partial^\mu + m^2\big)\delta^{(\mathsf{d})}(x-y)\phi(y) \\ &=: \int \mathrm{d}^{\mathsf{d}}x\, \mathrm{d}^{\mathsf{d}}y\, \phi(x)K(x-y)\phi(y)\,,\end{aligned} \tag{3.2.7}$$

where $K(x-y) = \big(\partial_\mu\partial^\mu + m^2\big)\delta^{(\mathsf{d})}(x-y)$ is the integral kernel for the real scalar field theory, and we define the integral operator,

$$\hat{K}\cdot\phi(x) = \int \mathrm{d}^{\mathsf{d}}x\, K(x-y)\phi(y)\,. \tag{3.2.8}$$

Then, formally applying the formula (3.2.6) to the infinite dimensional case, we obtain

$$\begin{aligned}Z_0[J] &= \int D\phi\, \exp\left(\mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\, \mathrm{d}^{\mathsf{d}}y\left(-\frac{1}{2}\phi(x)K(x-y)\phi(y)\right) + \mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\, J(x)\phi(x)\right) \\ &= Z_0[J=0]\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^{\mathsf{d}}x\, \mathrm{d}^{\mathsf{d}}y\, J(x)K^{-1}(x-y)J(y)\right).\end{aligned} \tag{3.2.9}$$

Here $K^{-1}(x-y)$ is defined as the inverse of the integral kernel $K(x-y)$,

$$\int \mathrm{d}^{\mathsf{d}}y\, K(x-y)K^{-1}(y-z) = (\partial_\mu\partial^\mu + m^2)K^{-1}(x-z) = \delta^{(\mathsf{d})}(x-z)\,, \tag{3.2.10}$$

which is interpreted as the corresponding **Green's function**, also called the **propagator**. Let $(\lambda)_{i\in\mathbb{Z}_+}$ be the eigenvalues of the kernel (the spectra of the Klein–Gordon operator), s.t.,

$$\mathrm{i}\hat{K}\cdot\phi_i(x) = \mathrm{i}\int \mathrm{d}^{\mathsf{d}}y\, K(x-y)\phi_i(y) = \lambda_i\,\phi_i(x)\,. \tag{3.2.11}$$

Then, the $J=0$ part of the partition function $Z[J=0]$ is given as the functional (Laplacian) determinant as follows,[2]

$$Z_0[J=0] \propto \det\left(\mathrm{i}\hat{K}\right)^{-1/2} = \prod_{i=1}^{\infty}\frac{1}{\sqrt{\lambda_i}}\,. \tag{3.2.14}$$

Applying the functional derivative formula (3.1.7) to the generating functional (3.2.9), we see that the 2-point function for the free scalar theory is given by Green's function:

$$\langle \mathrm{T}[\phi(x)\phi(y)]\rangle_0 = \frac{-1}{Z_0[J]}\left.\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)}\right|_{J\to 0} = -\mathrm{i}K^{-1}(x-y)\,. \tag{3.2.15}$$

Exercise 3.3 (Wick's theorem)**.** *The $2n$-point function is given by summation over the pair contributions,*

$$\begin{aligned}\langle \mathrm{T}[\phi(x_1)\cdots\phi(x_{2n})]\rangle_0 &= \sum_{\textit{possible pairs}}(-\mathrm{i})^n K^{-1}(x_{i_1}-x_{i_2})\cdots K^{-1}(x_{i_{2n-1}}-x_{i_{2n}}) \\ &= \operatorname*{Hf}_{1\le i,j\le 2n}\left(-\mathrm{i}K^{-1}(x_i-x_j)\right)\,, \end{aligned} \tag{3.2.16}$$

where we define the ***Hafnian*** *for the size $2n$ symmetric matrix, $A_{ji}=A_{ij}$,*

$$\mathrm{Hf}\,A = \frac{1}{2^n n!}\sum_{\sigma\in\mathfrak{S}_{2n}}\prod_{i=1}^{n}A_{\sigma(2i-1)\sigma(2i)}\,. \tag{3.2.17}$$

We denote the symmetric group of the rank $2n$ by $\mathfrak{S}_{2n}$.

1. *Verify this formula for the 4-point function,*

$$\begin{aligned}\langle \mathrm{T}[\phi(x_1)\cdots\phi(x_4)]\rangle_0 = &-K^{-1}(x_1-x_2)K^{-1}(x_3-x_4) - K^{-1}(x_1-x_3)K^{-1}(x_2-x_4) \\ &- K^{-1}(x_1-x_4)K^{-1}(x_2-x_3)\,. \end{aligned} \tag{3.2.18}$$

2. *Show that the n-point function becomes zero for $n\in 2\mathbb{Z}+1$,*

$$\langle \mathrm{T}[\phi(x_1)\cdots\phi(x_n)]\rangle_0 = 0 \qquad \textit{for} \qquad n\in 2\mathbb{Z}+1\,. \tag{3.2.19}$$

[2] We may regularize this infinite product by introducing the spectral zeta function,

$$\zeta(s) = \sum_{i=1}^{\infty}\frac{1}{\lambda_i^s} \implies \left.\frac{\partial}{\partial s}\right|_{s=0}\zeta(s) = -\sum_{i=1}^{\infty}\log\lambda_i\,. \tag{3.2.12}$$

Precisely speaking, the infinite sum (Dirichlet series) converges for sufficiently large s, so that we should properly consider the analytic continuation. Then, we obtain

$$Z_0[J=0] = \exp\left(\frac{1}{2}\left.\frac{\partial}{\partial s}\right|_{s=0}\zeta(s)\right)\,. \tag{3.2.13}$$

This is called the **zeta function regularization** scheme. In order to apply this regularization scheme, we assume that the Dirac spectrum is discrete (the spacetime is compact or the field ϕ is well localized due to the potential).

The formula (3.2.16) is called **Wick's theorem** for the scalar field, which is graphically expressed as follows:

$$\langle \mathrm{T}[\phi(x_1)\phi(x_2)]\rangle_0 = \wick{\c\phi_1\c\phi_2} = -\mathrm{i}K_{12}^{-1}\,, \tag{3.2.20a}$$

$$\begin{aligned}\langle \mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle_0 &= \wick{\c\phi_1\c\phi_2\c\phi_3\c\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \\ &= -K_{12}^{-1}K_{34}^{-1} - K_{13}^{-1}K_{24}^{-1} - K_{14}^{-1}K_{23}^{-1}\,,\end{aligned} \tag{3.2.20b}$$

where we denote $\phi_i = \phi(x_i)$ and $K_{ij}^{-1} = K^{-1}(x_i - x_j)$.

Exercise 3.4 (Momentum space representation)**.** *The integral representation of the delta function is given as*

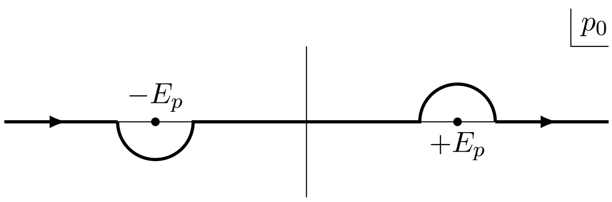
$$\delta^{(\mathsf{d})}(x) = \int \frac{\mathrm{d}^{\mathsf{d}}p}{(2\pi)^{\mathsf{d}}}\, \mathrm{e}^{-\mathrm{i}p\cdot x}\,. \tag{3.2.21}$$

1. *Show that the propagator given by*

$$\mathrm{i}\,\langle \mathrm{T}[\phi(x)\phi(y)]\rangle_0 = K^{-1}(x-y) = -\int \frac{\mathrm{d}^{\mathsf{d}}p}{(2\pi)^{\mathsf{d}}} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.2.22}$$

obeys the relation (3.2.10)*.*

2. *The parameter ε is introduced to avoid the poles of the integrand at $p^2 = m^2$. Denoting $p = (p_0, \vec{p})$, these poles are at $p_0^2 = \vec{p}^2 + m^2 =: E_p^2$. We shift the pole by $p_0 = \pm(E_p - \mathrm{i}\varepsilon)$, which is equivalent to taking the p_0-integral contour*



(3.2.23)

Then, verify that it reproduces the time-ordering product,

$$\langle \mathrm{T}[\phi(x)\phi(y)]\rangle_0 = \begin{cases} \langle \phi(x)\phi(y)\rangle_0 & (x_0 > y_0) \\ \langle \phi(y)\phi(x)\rangle_0 & (x_0 < y_0) \end{cases} \tag{3.2.24}$$

where the two-point function is given by

$$\langle \phi(x)\phi(y)\rangle_0 = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_p}\,. \tag{3.2.25}$$

3.3 Free Dirac field

We then turn to the Dirac field with the Lagrangian,

$$\mathscr{L}_0 = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi\,. \tag{3.3.1}$$

The corresponding generating functional is given by

$$Z_0[\bar{\eta},\eta] = \int D\bar{\psi}D\psi\, \exp\left(\mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\, \left(\bar{\psi}(x)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right)\right), \tag{3.3.2}$$

where $(\eta(x), \bar{\eta}(x))$ are the external spinor source fields. In the path integral formalism, the spinor fields are described by Grassmann numbers.

Exercise 3.5 (Gaussian integral of the Grassmann variables)**.** *Define the* ***Pfaffian*** *for a skew-symmetric matrix of size* $2n$*,* $A_{ij} = -A_{ji}$*,*

$$\mathrm{Pf}\, A = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} (-1)^\sigma \prod_{i=1}^{n} A_{\sigma(2i-1)\sigma(2i)} \,, \tag{3.3.3}$$

where $\mathfrak{S}_{2n}$ *is the symmetric group of the rank* $2n$*, and* $(-1)^\sigma$ *is the signature of the permutation* σ*.*[3] *The square of the Pfaffian is given by the determinant,*

$$(\mathrm{Pf}\, A)^2 = \det A \,. \tag{3.3.4}$$

1. *Derive the expressions of the Pfaffian for* $n = 1, 2$*,*

$$\mathrm{Pf}\, A = A_{12} \qquad (n = 1) \tag{3.3.5a}$$

$$= A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} \qquad (n = 2) \tag{3.3.5b}$$

   and verify the relation (3.3.4) *for these cases.*

2. *Based on the integrals of Grassmann numbers defined in Sec. 2.5.4, verify the Gaussian integral formula for* $n = 1, 2$*,*

$$\int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \exp\left(-\frac{1}{2} \sum_{i,j=1}^{2n} \theta_i A_{ij} \theta_j \right) = \mathrm{Pf}\, A \,. \tag{3.3.6}$$

3. *Using the formula* (3.3.6)*, show the complex version of the Gaussian integral*

$$\int \mathrm{d}\theta_1\, \mathrm{d}\theta_1^* \cdots \mathrm{d}\theta_n\, \mathrm{d}\theta_n^* \exp\left(- \sum_{i,j=1}^{n} \theta_i^* A_{ij} \theta_j \right) = \det A \,. \tag{3.3.7}$$

   We remark that there is no restriction to the matrix size for the complex case.

Using the Gaussian integral formulas, the generating functional is given as follows:

$$\frac{Z_0[\bar{\eta}, \eta]}{Z_0[\bar{\eta}, \eta = 0]} = \exp\left(-\mathrm{i} \int \mathrm{d}^\mathsf{d} x\, \mathrm{d}^\mathsf{d} y\, \bar{\eta}(x) K_\mathrm{D}^{-1}(x - y) \eta(y) \right) , \tag{3.3.8}$$

where Green's function is defined as the inverse of the integral kernel for the Dirac field, also called the **Dirac operator**,

$$K_\mathrm{D}(x - y) = (i\gamma^\mu \partial_\mu - m)\delta^{(\mathsf{d})}(x - y) \,. \tag{3.3.9}$$

The zero-source part of the partition function is given by the functional determinant similarly to (3.2.14),

$$Z_0[\bar{\eta}, \eta = 0] = \det(\mathrm{i}K_\mathrm{D}) \,. \tag{3.3.10}$$

Starting with the Majorana spinor, instead of the Dirac spinor, the partition function is replaced with the Pfaffian of the Dirac operator, $\mathrm{Pf}(\mathrm{i}K_\mathrm{D})$.

[3] We remark the similarity with the Hafnian (3.2.17).

Correlation function

We can compute the correlation function similarly to the scalar field theory. The 2-point function is given by

$$\left\langle \mathrm{T}\left[\psi(x)\bar{\psi}(y)\right]\right\rangle_0 = \mathrm{i}K_{\mathrm{D}}^{-1}(x-y)\,. \tag{3.3.11}$$

We remark that, since the spinor field consists of several components, $\psi = (\psi_\alpha)$, the 2-point function is a matrix-valued function as well as the Dirac operator K_{D}. In general, n-point function is given by Wick's theorem:

$$\left\langle \mathrm{T}\left[\psi(x_1)\bar{\psi}(x_2)\right]\right\rangle_0 = \wick{\c \psi_1 \c {\bar\psi}_2} = \mathrm{i}K_{12}^{-1}\,, \tag{3.3.12a}$$

$$\begin{aligned}\left\langle \mathrm{T}\left[\psi(x_1)\bar{\psi}(x_2)\psi(x_3)\bar{\psi}(x_4)\right]\right\rangle_0 &= \wick{\c \psi_1 \c {\bar\psi}_2 \c \psi_3 \c {\bar\psi}_4} + \wick{\c2 \psi_1 \c1 {\bar\psi}_2 \c1 \psi_3 \c2 {\bar\psi}_4} \\ &= -K_{12}^{-1}K_{34}^{-1} - K_{14}^{-1}K_{23}^{-1}\,,\end{aligned} \tag{3.3.12b}$$

where $K_{ij}^{-1} = K_{\mathrm{D}}^{-1}(x_i - x_j)$. In this case, the contraction between ψ_i and ψ_j ($\bar{\psi}_i$ and $\bar{\psi}_j$), $\wick{\c \psi_i \c \psi_j}$, does not contribute to the correlation function.

3.4 Interacting field

We consider the interacting field theory based on the free field description discussed above. We start with the following identity,

$$f\left(-\mathrm{i}\frac{\delta}{\delta J}\right)Z_0[J] = \int D\phi\, f(\phi)\exp\left(\mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\,(\mathscr{L}_0(\phi,\partial\phi) + J(x)\phi(x))\right). \tag{3.4.1}$$

Applying this identity, we obtain the generating functional in the presence of the interaction term $\mathscr{L}_{\mathrm{int}}(\phi)$,[4]

$$\begin{aligned}Z[J] &= \int D\phi\,\exp\left(\mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\,(\mathscr{L}_0(\phi,\partial\phi) + \mathscr{L}_{\mathrm{int}}(\phi) + J(x)\phi(x))\right) \\ &= \exp\left(\mathrm{i}\int \mathrm{d}^{\mathsf{d}}x\left(\mathscr{L}_{\mathrm{int}}\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)\right)\right)Z_0[J]\,.\end{aligned} \tag{3.4.2}$$

In the operator formalism, this expression corresponds to the **interaction picture**, while the previous one (3.1.5) is based on the **Heisenberg picture**.

In order to apply the perturbative expansion, we discuss an alternative expression of the generating function. Recalling the expression for the free part of the generating function (3.2.9),

[4]We assume that the interaction part of the Lagrangian does not contain the derivative terms, $\partial\phi(x)$, for simplicity. In general, we can also incorporate the derivative terms. See, for example, Chapter 5.

the full generating function is given by

$$\begin{aligned}\frac{Z[J]}{Z_0[J=0]} &= \exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\left(\mathscr{L}_{\mathrm{int}}\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)\right)\right)\frac{Z_0[J]}{Z_0[J=0]} \\ &= \exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\left(\mathscr{L}_{\mathrm{int}}\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)\right)\right)\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^\mathsf{d}x\,\mathrm{d}^\mathsf{d}y\,J(x)K^{-1}(x-y)J(y)\right) \\ &= \exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\left(\mathscr{L}_{\mathrm{int}}\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)\right)\right) \\ &\qquad \times \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^\mathsf{d}x\,\mathrm{d}^\mathsf{d}y\,\frac{\delta}{\delta\phi(x)}K^{-1}(x-y)\frac{\delta}{\delta\phi(y)}\right)\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,J(x)\phi(x)\right)\Bigg|_{\phi=0} \\ &= \exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^\mathsf{d}x\,\mathrm{d}^\mathsf{d}y\,\frac{\delta}{\delta\phi(x)}K^{-1}(x-y)\frac{\delta}{\delta\phi(y)}\right) \\ &\qquad \times \exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,\mathscr{L}_{\mathrm{int}}(\phi)\right)\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,J(x)\phi(x)\right)\Bigg|_{\phi=0} \end{aligned} \tag{3.4.3}$$

Exercise 3.6. *Verify the formula*

$$\exp\left(-\frac{\mathrm{i}}{2}\int \mathrm{d}^\mathsf{d}x\,\mathrm{d}^\mathsf{d}y\,\frac{\delta}{\delta\phi(x)}K^{-1}(x-y)\frac{\delta}{\delta\phi(y)}\right)\mathscr{O}(\phi)\Bigg|_{\phi=0} = \langle \mathrm{T}[\mathscr{O}(\phi)]\rangle_0 \tag{3.4.4}$$

for

$$\mathscr{O} = \phi(x_1)\cdots\phi(x_n)\,, \tag{3.4.5}$$

which reproduces the n*-point function* (3.2.16)*.*

With this formula, we obtain the following expression for the generating function,

$$\frac{Z[J]}{Z_0[J=0]} = \left\langle \mathrm{T}\left[\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,(\mathscr{L}_{\mathrm{int}}(\phi)+J(x)\phi(x))\right)\right]\right\rangle_0\,, \tag{3.4.6a}$$

$$\frac{Z[J]}{Z[J=0]} = \left\langle \mathrm{T}\left[\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,(\mathscr{L}_{\mathrm{int}}(\phi)+J(x)\phi(x))\right)\right]\right\rangle_0 \Big/ \left\langle \mathrm{T}\left[\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,\mathscr{L}_{\mathrm{int}}(\phi)\right)\right]\right\rangle_0\,. \tag{3.4.6b}$$

The reason why we normalize it by $Z[J=0]$ is to subtract the vacuum contributions. If the interaction part of the Lagrangian contribution is sufficiently small, one can evaluate the generating function with the following expansion[5]

$$\begin{aligned}&\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,(\mathscr{L}_{\mathrm{int}}(\phi)+J(x)\phi(x))\right) \\ &= \sum_{m=0}^{\infty}\frac{\mathrm{i}^m}{m!}\int \mathrm{d}^\mathsf{d}x_1\cdots\mathrm{d}^\mathsf{d}x_m\,\mathscr{L}_{\mathrm{int}}(\phi(x_1))\cdots\mathscr{L}_{\mathrm{int}}(\phi(x_m))\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,J(x)\phi(x)\right)\,.\end{aligned} \tag{3.4.7}$$

We can apply the same analysis for the Dirac field,

$$\begin{aligned}\frac{Z[\eta,\bar\eta]}{Z_0[\eta,\bar\eta=0]} &= \exp\left(-\mathrm{i}\int \mathrm{d}^\mathsf{d}x\,\mathrm{d}^\mathsf{d}y\,\frac{\delta}{\delta\psi(x)}K_{\mathrm{D}}^{-1}(x-y)\frac{\delta}{\delta\bar\psi(y)}\right) \\ &\qquad \times\exp\left(\mathrm{i}\int \mathrm{d}^\mathsf{d}x\left(\mathscr{L}_{\mathrm{int}}\left(\frac{\delta}{\delta\psi},\frac{\delta}{\delta\bar\psi}\right)+\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)\right)\right)\Bigg|_{\psi,\bar\psi=0}\,.\end{aligned} \tag{3.4.8}$$

This is a starting point of the perturbative analysis for the interacting field theory.

[5] Although this gives an asymptotic expansion, which does not converge in general, one can obtain very high-precision results for several theories.

3.5 Feynman rule

3.5.1 Scalar field theory

Let us demonstrate the perturbative expansion (3.4.7) with the ϕ^4-theory (2.4.1), as an example. In this case, the interaction term is given by

$$\mathscr{L}_{\text{int}}(\phi) = -\frac{\lambda}{4!}\phi(x)^4 \,. \tag{3.5.1}$$

The parameter λ is called the **coupling constant**, which characterizes the self-interaction (nonlinearity) effect. From this point of view, (3.4.7) is an asymptotic expansion with respect to the coupling constant λ, so that we should assume that it is sufficiently small.

For example, the two-point correlation function is given by the functional derivative of the generating function,

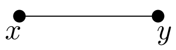
$$\begin{aligned}\langle \mathrm{T}[\phi(z_1)\phi(z_2)]\rangle &= \frac{-1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(z_1)\delta J(z_2)}\bigg|_{J\to 0} \\ &\overset{(3.4.7)}{=} \sum_{m=0}^{\infty}\frac{\mathrm{i}^m}{m!}\int \mathrm{d}^{\mathsf{d}}x_1\cdots\mathrm{d}^{\mathsf{d}}x_m \,\langle \mathrm{T}[\mathscr{L}_{\text{int}}(x_1)\cdots\mathscr{L}_{\text{int}}(x_m)\times\phi(z_1)\phi(z_2)]\rangle_0 \\ &= \sum_{m=0}^{\infty}\frac{(-\mathrm{i}\lambda)^m}{(4!)^m m!}\int \mathrm{d}^{\mathsf{d}}x_1\cdots\mathrm{d}^{\mathsf{d}}x_m \left\langle \mathrm{T}\left[\phi(x_1)^4\cdots\phi(x_m)^4\phi(z_1)\phi(z_2)\right]\right\rangle_0 \,.\end{aligned} \tag{3.5.2}$$

This shows the expansion of the full correlation function with the bare correlation function (the correlation function of the free theory). This can be evaluated with Wick's theorem discussed in Sec. 3.2. The $O(\lambda)$ term is given as follows:

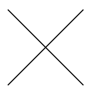
$$\left\langle \mathrm{T}\left[\phi^4\cdot\phi_1\phi_2\right]\right\rangle_0 = \phi\phi\phi\phi\cdot\phi_1\phi_2 + \phi\phi\phi\phi\cdot\phi_1\phi_2 \,. \tag{3.5.3}$$

We see that there are two kinds of contributions to the two-point function. One can apply this formalism similarly to higher-point functions, but it seems more involved.[6]

Instead of applying Wick's theorem, we introduce the **Feynman rule** to evaluate the correlation function:

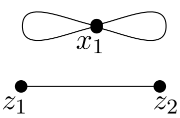
Propagator: 

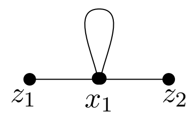
$= -\mathrm{i}K^{-1}(x-y)$ (3.5.4a)

Vertex: 

$= -\mathrm{i}\lambda\int \mathrm{d}^{\mathsf{d}}x$ (3.5.4b)

In this formalism, the $O(\lambda)$ contribution (3.5.3) is graphically given by (but we do not yet integrate the variable x_1)

$\left\langle \mathrm{T}\left[\phi(x_1)^4\cdot\phi(z_1)\phi(z_2)\right]\right\rangle_0 =$ 

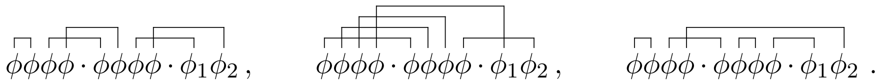
$+$ 

. (3.5.5)

Then, apparently the first one is a disconnected diagram, which will be subtracted in the end. The second term has an interpretation as an amplitude from z_1 to z_2 with interaction at the point x_1, which will be integrated out to take into account all the possible configurations like this.

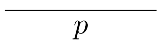
[6] The $O(\lambda^2)$ contribution is given by ten field insertion in total, $\left\langle \mathrm{T}\left[\phi(x_1)^4\cdot\phi(x_2)^4\cdot\phi(z_1)\phi(z_2)\right]\right\rangle_0$.

Exercise 3.7 (Two-loop contribution)**.** *Draw the Feynman diagrams appearing in the* $O(\lambda^2)$ *contribution* $\left\langle \mathrm{T}\left[\phi(x_1)^4 \cdot \phi(x_2)^4 \cdot \phi(z_1)\phi(z_2)\right]\right\rangle_0$*, which correspond to the contractions,*

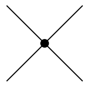


(3.5.6)

One can similarly formulate the Feynman rule in the momentum space. In this case, we should impose the momentum conservation at each vertex.

Propagator: 

$$= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.5.7a}$$

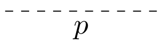
Vertex: 
$$= -\mathrm{i}\lambda \tag{3.5.7b}$$

3.5.2 Dirac field theory

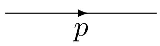
Let us consider the perturbative expansion for the fermionic field theory. We now consider the Yukawa interaction as an example,

$$\mathscr{L}_{\text{int}} = -g\bar{\psi}\psi\phi\,. \tag{3.5.8}$$

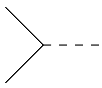
Since we have both the scalar field and the Dirac field in this case, we have accordingly two propagators,

Bosonic propagator: 

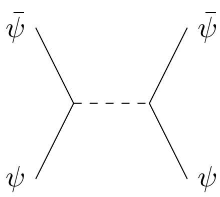
$$= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.5.9a}$$

Dirac propagator: 

$$= \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon} \tag{3.5.9b}$$

Vertex: 
$$= -\mathrm{i}g \tag{3.5.9c}$$

In this case, the fermionic propagator is oriented. For example, the two-particle scattering process is described by



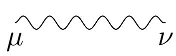
(3.5.10)

Identifying the scalar field as a scalar meson, the Dirac field as quarks, such a process is interpreted as meson exchange process, which plays an important role in the nuclear interaction.

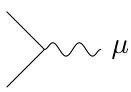
Another example is the electromagnetic interaction,

$$\mathscr{L}_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu\,, \tag{3.5.11}$$

with the diagrams,

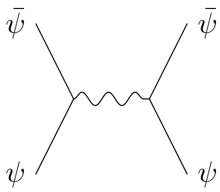
Photon propagator: 

$$= \frac{\mathrm{i}g_{\mu\nu}}{p^2 + \mathrm{i}\varepsilon} \tag{3.5.12a}$$

Vertex: 

$$= \mathrm{i}e\gamma_\mu \tag{3.5.12b}$$

Since the photon field A_μ has the vector index μ, we should specify it in the diagram. For example, one can derive the Coulomb interaction between the electrons from the diagram in the non-relativistic limit:



(3.5.13)

Exercise 3.8 (Fermion and photon propagators)**.**

1. *Derive the momentum space representation of the fermion propagator* (3.5.9b) *from the Dirac Lagrangian* (2.5.15)*.*

2. *Derive the photon propagator* (3.5.12a) *in the Lorentz gauge* $\partial_\mu A^\mu = 0$ *from the Maxwell Lagrangian* (2.6.13)*.*

3.6 Effective action

The generating functional (3.1.5) is a functional depending on the external field $J(x)$. We introduce another functional, that is called the **effective action** from the Legendre transform of the generating functional for the connected part (3.1.10),

$$\Gamma[\phi] = W[J] - \int \mathrm{d}^\mathsf{d} x\, J(x)\phi(x)\,. \tag{3.6.1}$$

Recalling the definition of the generating functional (3.1.5), $\Gamma[\phi]$ agrees with the action $S[\phi]$ in the classical limit,

$$Z[J] = \mathrm{e}^{\mathrm{i}W[J]} \approx \mathrm{e}^{\mathrm{i}S[\phi] + \mathrm{i}\int J(x)\phi(x)}\,. \tag{3.6.2}$$

From the functional derivatives of these functional, we obtain

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x)\,, \qquad \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x)\,. \tag{3.6.3}$$

Since we will take the limit $J \to 0$, we will obtain $\delta\Gamma[\phi]/\delta\phi(x) = 0$, which is interpreted as an "effective" version of the Euler–Lagrange equation. The second derivatives are

$$\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = \frac{\delta\phi(x)}{\delta J(y)} = \frac{\delta\phi(y)}{\delta J(x)}\,, \qquad \frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\delta\phi(y)} = -\frac{\delta J(x)}{\delta\phi(y)} = -\frac{\delta J(y)}{\delta\phi(x)} =: X(x,y)\,, \tag{3.6.4}$$

which yield the relation

$$\int \mathrm{d}^\mathsf{d} y\, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta^2\Gamma[\phi]}{\delta\phi(y)\delta\phi(z)} = -\int \mathrm{d}^\mathsf{d} y\, \frac{\delta\phi(x)}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(z)} = -\delta^{(\mathsf{d})}(x-z)\,. \tag{3.6.5}$$

Namely, the functional Hessian of the effective action is the inverse of that of the generating functional, which gives rise to the connected two-point function in the presence of the external source,

$$-\frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} = X^{-1}(x,y) = \langle \mathrm{T}[\phi(x)\phi(y)]\rangle_{\mathrm{c},J} \tag{3.6.6}$$

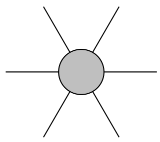
In order to understand the role of the effective action, we further take a functional derivative of (3.6.5) with the external source $J(w)$, which gives rise to

$$\begin{aligned}0 &= \int \mathrm{d}^{\mathsf{d}}y\, \frac{\delta^3 W[J]}{\delta J(x)\delta J(y)\delta J(w)} \frac{\delta^2 \Gamma[\phi]}{\delta\phi(y)\delta\phi(z)} \\ &\quad + \int \mathrm{d}^{\mathsf{d}}y\, \mathrm{d}^{\mathsf{d}}w'\, \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)} \frac{\delta^3 \Gamma[\phi]}{\delta\phi(y)\delta\phi(z)\delta\phi(w')} \frac{\delta\phi(w')}{\delta J(w)} \\ &= \int \mathrm{d}^{\mathsf{d}}y\, \frac{\delta^3 W[J]}{\delta J(x)\delta J(y)\delta J(w)} X(y,z) \\ &\quad + \int \mathrm{d}^{\mathsf{d}}y\, \mathrm{d}^{\mathsf{d}}w'\, \big(-X^{-1}(x,y)\big)\big(-X^{-1}(w',w)\big) \frac{\delta^3 \Gamma[\phi]}{\delta\phi(y)\delta\phi(z)\delta\phi(w')} \,. \end{aligned} \tag{3.6.7}$$

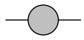
Therefore, we obtain

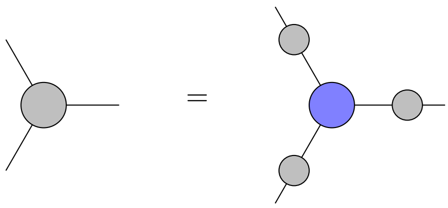
$$\begin{aligned}&\frac{\delta^3 W[J]}{\delta J(x)\delta J(y)\delta J(z)} \\ &= \int \mathrm{d}^{\mathsf{d}}w\, \mathrm{d}^{\mathsf{d}}w'\, \mathrm{d}^{\mathsf{d}}w''\, \big(-X^{-1}(x,w)\big)\big(-X^{-1}(y,w')\big)\big(-X^{-1}(z,w'')\big) \frac{\delta^3 \Gamma[\phi]}{\delta\phi(w)\delta\phi(w')\delta\phi(w'')} \,. \end{aligned} \tag{3.6.8}$$

We denote the connected n-point function by



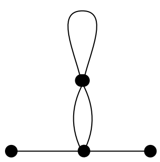
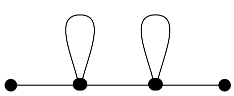
(n-legs) . (3.6.9)

We remark the connected two-point function is given by X^{-1} = . Then, the relation (3.6.8) implies that the connected three-point function has the following decomposition:

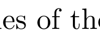


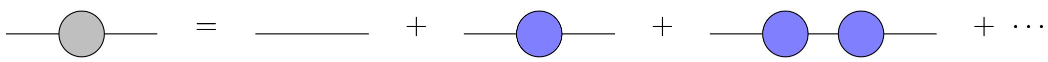
(3.6.10)

where the blue symbol shows the **one-particle irreducible (1PI) diagram**, which cannot be split into two diagrams by removing a single line. For example, we have

 $\in$ 1PI  $\notin$ 1PI (3.6.11)

Exercise 3.9. *Show that the effective action is a generating function of the 1PI n-point functions by induction (We already show it is true for $n = 3$).*

In particular, the 1PI two-point function $-\mathrm{i}\Sigma :=$  is called the **self-energy**. In fact, the connected two-point function is given by the series of the self energies:



(3.6.12)

In the case of ϕ^4-theory, we have $\langle \phi(x_1)\phi(x_2)\rangle_{\mathrm{c}} = \langle \phi(x_1)\phi(x_2)\rangle$ (see Sec. 3.1). Hence, in the momentum space, this series is given for the scalar field theory (3.5.7a) by

$$\begin{aligned} \int \mathrm{d}^{\mathsf{d}}x \, \mathrm{e}^{\mathrm{i}p\cdot x} \langle \mathrm{T}[\phi(x)\phi(0)] \rangle &= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} + \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \big(-\mathrm{i}\Sigma(p^2) \big) \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} + \cdots \\ &= \frac{\mathrm{i}}{p^2 - m^2 - \Sigma(p^2) + \mathrm{i}\varepsilon} \, . \end{aligned} \tag{3.6.13}$$

Namely, the self-energy gives the shift of the mass parameter. A similar expression is also possible for the Dirac field propagator.

Chapter 4

Loop correction and renormalization

We have introduced the path integral formalism to quantize the field theories. In this Chapter, we actually calculate the correlation functions and see how to deal with the diverging integrals often appearing in the calculation. We then introduce the notion of the renormalization and discuss its physical consequences.

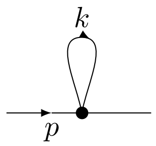
4.1 Scalar field theory

We consider the scalar field theory to demonstrate how to calculate the correlation function. The spinor field would be treated in a similar manner.

4.1.1 Two-point function

As discussed in (3.6.13), we need to evaluate the self-energy Σ to compute the connected two-point function. At the one-loop order, we have a single contribution

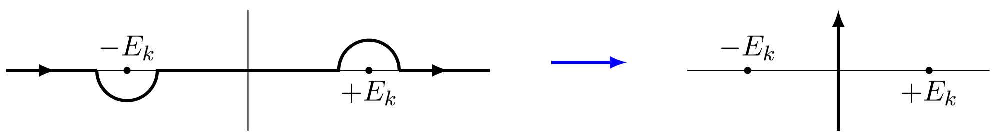
$$-\mathrm{i}\Sigma(p^2) =$$



$$= \frac{-\mathrm{i}\lambda}{2} \int \frac{\mathrm{d}^{\mathsf{d}} k}{(2\pi)^{\mathsf{d}}} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \, . \tag{4.1.1}$$

The factor 2 is due to the symmetry of the diagram. We remark that this expression does not depend on the external momentum p, which is a specific feature of the one-loop diagram. In general, the self-energy depends on the external momentum.

In order to evaluate this integral, we change the contour (3.2.23) as follows:



(4.1.2)

This contour change corresponds to $k_0 \to \mathrm{i}k_0$, therefore $k^2 = k_0^2 - \vec{k}^2 \to -k_0^2 - \vec{k}^2 =: -k_{\mathrm{E}}^2$, which is the (minus of) norm with the Euclidean signature. This prescription is called the **Wick rotation**. Hence, we obtain

$$\Sigma(p^2) = \frac{\lambda}{2} \int \frac{\mathrm{d}^{\mathsf{d}} k_{\mathrm{E}}}{(2\pi)^{\mathsf{d}}} \frac{1}{k_{\mathrm{E}}^2 + m^2} = \frac{\lambda}{2} \int_0^\infty \mathrm{d}k \, \frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}} \frac{k^{\mathsf{d}-1}}{k^2 + m^2} \, , \tag{4.1.3}$$

where $\mathrm{d}^{\mathsf{d}}k_{\mathrm{E}} = \Omega_{\mathsf{d}-1} k^{\mathsf{d}-1}\,\mathrm{d}k$ with Ω_{d} the volume of d-sphere S^{d} of radius one.

Exercise 4.1 (Dimensional regularization)**.**

1. *Derive the volume of* d*-sphere* S^{d} *of radius one,*

$$\Omega_{\mathsf{d}-1} = \frac{2\pi^{\mathsf{d}/2}}{\Gamma(\mathsf{d}/2)}\,, \tag{4.1.4}$$

so that

$$\Omega_1 = 2\pi\,, \qquad \Omega_2 = 4\pi\,, \qquad \Omega_3 = 2\pi^2\,, \quad \textit{etc.} \tag{4.1.5}$$

Hint. One may derive it from the relation

$$\int_{\mathbb{R}^{\mathsf{d}}} \mathrm{d}^{\mathsf{d}}x\, \mathrm{e}^{-(x_1^2+\cdots+x_{\mathsf{d}}^2)^2} = \int_{\mathbb{R}^{\mathsf{d}}} \mathrm{d}^{\mathsf{d}}x\, \mathrm{e}^{-r^2} = \Omega_{\mathsf{d}-1}\int_0^\infty \mathrm{d}r\, r^{\mathsf{d}-1}\mathrm{e}^{-r^2} \tag{4.1.6}$$

together with the Gaussian integral (3.2.3) *and the integral formula of the gamma function*

$$\Gamma(s) = \int_0^\infty \mathrm{d}t\, t^{s-1}\mathrm{e}^{-t}\,. \tag{4.1.7}$$

2. *Derive the formula*

$$\int_0^\infty \mathrm{d}k\, \frac{k^\alpha}{(k^2+\Delta^2)^\beta} = \frac{\Delta^{\alpha-2\beta+1}}{2}\frac{\Gamma(-\alpha/2+\beta-1/2)\Gamma(\alpha/2+1/2)}{\Gamma(\beta)}\,. \tag{4.1.8}$$

Hint. Use the variable $x = \Delta^2/(k^2+\Delta^2) \in [0,1]$ *and the beta integral formula*

$$B(a,b) = \int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{4.1.9}$$

3. *Derive the one-loop expression of the self-energy,*

$$\Sigma(p^2) = \frac{\lambda}{2}\frac{m^{\mathsf{d}-2}}{(4\pi)^{\mathsf{d}/2}}\Gamma\left(1-\frac{\mathsf{d}}{2}\right)\,. \tag{4.1.10}$$

As discussed in Sec. 1.3.3, the mass dimension of the coupling constant is $[\lambda] = 4-\mathsf{d}$, so that we shift $\lambda \to \lambda\mu^{4-\mathsf{d}}$ with $[\mu] = 1$ in order that the coupling becomes dimensionless. Then, the self-energy becomes

$$\Sigma(p^2) = \frac{\lambda m^2}{2^{\mathsf{d}+1}\pi^{\mathsf{d}/2}}\left(\frac{\mu}{m}\right)^{4-\mathsf{d}}\Gamma\left(1-\frac{\mathsf{d}}{2}\right)\,. \tag{4.1.11}$$

This parameter μ will play an important role in the discussion of renormalization group in Sec. 4.3.

In fact, the gamma function $\Gamma(z)$ has poles at $-n \in \mathbb{Z}_{\le 0}$ with the residue

$$\operatorname*{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}\,, \tag{4.1.12}$$

so that the self-energy diverges for even dimensions larger than two. This behavior is now known to be an artifact of the dimensional regularization scheme, which only cares about the logarithmic divergence [Lei75, BM77]. In order to see this behavior, we consider more intuitive regularization with the parameter Λ, called the **cutoff**,

$$\Sigma(p^2) = \lim_{\Lambda\to\infty}\frac{\lambda}{2}\int_0^\Lambda \mathrm{d}k\, \frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}}\frac{k^{\mathsf{d}-1}}{k^2+m^2}\,. \tag{4.1.13}$$

We remark that the cutoff in the momentum space may be written using the lattice spacing discussed in Sec 1.3, $\Lambda \approx 2\pi/a$. This cutoff regularization is more intuitive than the dimensional regularization. However, it is not suitable for gauge theory since the dimensionfull parameter Λ violates the gauge symmetry, and thus the dimensional regularization seems better in this case.

Exercise 4.2 (Cutoff regularization)**.**

1. *Verify the expression*

$$\Sigma(p^2;\Lambda) := \frac{\lambda}{2}\int_0^\Lambda \mathrm{d}k \, \frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}} \frac{k^{\mathsf{d}-1}}{k^2+m^2} \overset{t=k^2/\Lambda^2}{=} \frac{\lambda\Lambda^{\mathsf{d}}}{4m^2}\frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}}\int_0^1 \mathrm{d}t \, t^{\mathsf{d}/2-1}\left(1+\frac{\Lambda^2}{m^2}t\right)^{-1} . \tag{4.1.14}$$

2. *Show*

$$\int_0^1 \mathrm{d}t \, t^{\mathsf{d}/2-1}(1+at)^{-1} = \begin{cases} a^{-1} - a^{-2}\log(1+a) & (\mathsf{d}=4) \\ 2a^{-1} - 2a^{-3/2}\tan^{-1}(a^{1/2}) & (\mathsf{d}=3) \\ a^{-1}\log(1+a) & (\mathsf{d}=2) \end{cases} \tag{4.1.15}$$

and evaluate the self-energy for $\mathsf{d} = 2,3,4$.

3. *Using the integral formula of the hypergeometric function*

$$\begin{aligned} {}_2F_1\left({a,b \atop c};z\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \mathrm{d}t \, t^{b-1}(1-x)^{c-b-1}(1-zt)^{-a} \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(1)_n(c)_n}z^n , \qquad (x)_n = \prod_{j=0}^{n-1}(x+j) , \end{aligned} \tag{4.1.16}$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$*, show that the cutoff dependent self-energy for generic* d *is given by*

$$\Sigma(p^2;\Lambda) = \frac{\lambda}{2\mathsf{d}}\frac{\Lambda^{\mathsf{d}}}{m^2}\frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}} \, {}_2F_1\left({1,\mathsf{d}/2 \atop \mathsf{d}/2+1};-\frac{\Lambda^2}{m^2}\right) . \tag{4.1.17}$$

4. *The convergence radius of the expansion* (4.1.16) *is* $|z|<1$*. Using (or deriving) the relation*

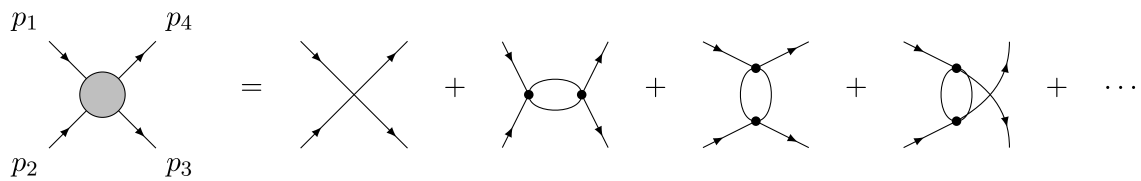
$$\begin{aligned} {}_2F_1\left({a,b \atop c};z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1\left({a,1+a-c \atop 1+a-b};z^{-1}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1\left({b,1+b-c \atop 1+b-a};z^{-1}\right) , \end{aligned} \tag{4.1.18}$$

expand the self-energy with the parameter $m^2/\Lambda^2 \ll 1$ *to show*

$$\Sigma(p^2;\Lambda) \approx \Lambda^{\mathsf{d}-2} + \textit{lower terms} . \tag{4.1.19}$$

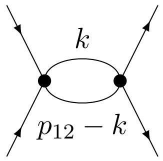
4.1.2 Four-point function

We consider the one-loop contribution to the four-point function:



(4.1.20)

The first one-loop diagram shows



$$= \frac{(-\mathrm{i}\lambda)^2}{2}\int \frac{\mathrm{d}^{\mathsf{d}}k}{(2\pi)^{\mathsf{d}}}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\varepsilon}\frac{\mathrm{i}}{(p_{12}-k)^2-m^2+\mathrm{i}\varepsilon}$$

$$= \frac{\mathrm{i}\lambda^2}{2}\int \frac{\mathrm{d}^{\mathsf{d}}k_{\mathrm{E}}}{(2\pi)^{\mathsf{d}}}\frac{1}{k_{\mathrm{E}}^2+m^2}\frac{1}{(p_{12}-k)_{\mathrm{E}}^2+m^2} =: \mathrm{i}\lambda^2 V_4(p_{12}^2) \tag{4.1.21}$$

where we denote $p_{12}=p_1+p_2$. Other diagrams are given by replacing p_{12} with $p_{13}=p_1-p_3$ and $p_{14}=p_1-p_4$, respectively. They are also denoted using the **Mandelstam variables**,

$$(p_{12}^2, p_{13}^2, p_{14}^2) = (s,t,u)\,. \tag{4.1.22}$$

Exercise 4.3 (Vertex function for the four-point function)**.**

1. *Show the following formulas:*

   (a) *Schwinger parametrization*

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}t\, t^{n-1}\mathrm{e}^{-At} \tag{4.1.23}$$

   (b) *Feynman parametrization*

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\, \frac{1}{(Ax+B(1-x))^2} \tag{4.1.24}$$

2. *Verify the formula*

$$\begin{aligned} V_4(p^2;\Lambda) &:= \frac{1}{2}\int_{|k_E|\le\Lambda}\frac{\mathrm{d}^{\mathsf{d}}k_E}{(2\pi)^{\mathsf{d}}}\frac{1}{k_E^2+m^2}\frac{1}{(p-k)_E^2+m^2} \\ &= \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^\Lambda \mathrm{d}^{\mathsf{d}}k_E\,\left(k_E^2+\Delta^2\right)^{-2} \qquad \left(\Delta^2 = m^2 + x(1-x)p^2\right) \\ &= \frac{\Lambda^{\mathsf{d}}}{4}\frac{\Omega_{\mathsf{d}-1}}{(2\pi)^{\mathsf{d}}}\int_0^1 \mathrm{d}x\,\Delta^{-4}\int_0^1 \mathrm{d}t\, t^{\mathsf{d}/2-1}\left(1+\frac{\Lambda^2}{\Delta^2}t\right)^{-2}, \end{aligned} \tag{4.1.25}$$

   and

$$\int_0^1 \mathrm{d}t\, t^{\mathsf{d}/2-1}(1+at)^{-2} = \begin{cases} a^{-2}\log(1+a) - a^{-1}(1+a)^{-1} & (\mathsf{d}=4) \\ a^{-3/2}\tan^{-1}(a^{1/2}) - a^{-1}(1+a)^{-1} & (\mathsf{d}=3) \\ (1+a)^{-1} & (\mathsf{d}=2) \end{cases} \tag{4.1.26}$$

   Then, derive the vertex function $V_4(p^2;\Lambda)$ for $\mathsf{d}=4$,

$$V_4(p^2;\Lambda) = \frac{1}{32\pi^2}\int_0^1 \mathrm{d}x \left[\log\left(1+\frac{\Lambda^2}{\Delta^2}\right) - \left(1+\frac{\Delta^2}{\Lambda^2}\right)^{-1}\right], \qquad (\mathsf{d}=4)\,. \tag{4.1.27}$$

   Similarly derive the expression for $\mathsf{d}=2,3$.

3. *Show that the vertex function (4.1.21) is given by*

$$V_4(p^2) = \frac{\Gamma(2-\mathsf{d}/2)}{2(4\pi)^{\mathsf{d}/2}}\int_0^1 \mathrm{d}x\,\Delta^{\mathsf{d}-4}\,. \tag{4.1.28}$$

   Hint. Use the variable $z=\Delta^2/(k^2+\Delta^2)\in[0,1]$ and apply the beta integral formula (4.1.9).

4. Using the infinite product formula of the gamma function

$$\Gamma(z)^{-1} = z\mathrm{e}^{\gamma z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n} \right) \mathrm{e}^{-z/n} \right) , \tag{4.1.29}$$

with γ Euler's constant, derive the expansion

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z) . \tag{4.1.30}$$

5. Let $\mathsf{d} = 4 - \epsilon$. Then, verify the formula

$$V_4(p^2) = \frac{1}{32\pi^2} \left[\frac{2}{\epsilon} - \gamma - \int_0^1 \mathrm{d}x \log \left(\frac{\Delta^2}{4\pi} \right) \right] . \tag{4.1.31}$$

4.2 Renormalization

We have seen various diverging behaviors of the loop integral. In order to provide a physical prediction from them, we have to regularize such divergences. The **renormalization** is a systematic scheme to extract a physically meaningful quantities from the diverging integrals. As seen in the previous computations, the mass and the coupling constant receive the quantum correction. Since we would observe the total quantities including all the corrections in experiments, the bare mass and the bare coupling constant appearing in the original Lagrangian are formal parameters, that would not be observed in the end. The idea of renormalization is to absorb the infinities by unobserved quantities.

4.2.1 Counter terms

For the ϕ^4 theory, we denote the bare and physical parameters by (m_0, λ_0) and (m, λ). We similarly denote the bare field by ϕ_0. Hence, the Lagrangian of this theory is given by

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{1}{4!}\lambda_0 \phi_0^4 . \tag{4.2.1}$$

We rewrite this Lagrangian in terms of the physical parameters and the rescaled field,

$$\phi_0 = Z_\phi^{1/2} \phi , \tag{4.2.2}$$

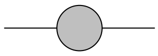
as follows:

$$\begin{aligned} \mathscr{L} &= \frac{1}{2} Z_\phi \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 Z_\phi \phi^2 - \frac{1}{4!} \lambda_0 Z_\phi^2 \phi^4 \\ &= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!}\lambda \phi^4 \\ &\quad + \frac{1}{2}\delta_{Z_\phi} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\delta_m \phi^2 - \frac{1}{4!}\delta_\lambda \phi^4 \end{aligned} \tag{4.2.3}$$

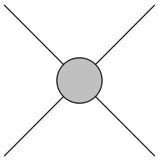
where the terms appearing in the third line are called the **counter terms** to absorb the infinities with

$$\delta_{Z_\phi} = Z_\phi - 1 , \qquad \delta_m = m_0^2 Z_\phi - m^2 , \qquad \delta_\lambda = \lambda_0 Z_\phi^2 - \lambda . \tag{4.2.4}$$

The physical mass is defined as the pole of the propagator, while there is no unique definition of the physical coupling constant. A standard one for the latter case is to use the four-point function with the momenta $p_i = (m,0,0,0)$ for $i = 1,\dots,4$, equivalently $s = 4m^2$, $t = u = 0$ in the Mandelstam variables (4.1.22),



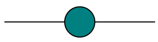
$$= \frac{\mathrm{i}}{p^2 - m^2} + \text{regular terms}\,, \tag{4.2.5a}$$



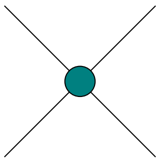
$$= -\mathrm{i}\lambda \quad \text{at} \quad (s,t,u) = (4m^2, 0, 0)\,. \tag{4.2.5b}$$

These give a **renormalization condition** for the mass and the coupling constant.

In addition to the propagator and the vertex (3.5.7), we introduce the Feynman diagrams for the counter terms,



$$= \mathrm{i}\big(p^2 \delta_{Z_\phi} - \delta_m\big) \tag{4.2.6a}$$



$$= -\mathrm{i}\delta_\lambda \tag{4.2.6b}$$

We demonstrate how to compute the two- and four-point functions with these diagrams in the following.

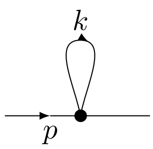
4.2.2 Two-point function

As shown before, we need to evaluate the self-energy to compute the two-point function. By definition of the physical mass, we obtain from (3.6.13) the renormalization condition,

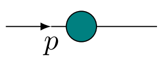
$$\Sigma(p^2)\Big|_{p^2=m^2} = 0\,, \qquad \frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\Bigg|_{p^2=m^2} = 0\,. \tag{4.2.7}$$

The one-loop contribution to the self-energy with the counter term is given by

$$-\mathrm{i}\Sigma(p^2) =$$



$$+$$



$$+ \quad \cdots$$

$$\begin{aligned} &= \frac{-\mathrm{i}\lambda}{2} \int \frac{\mathrm{d}^{\mathsf{d}}k}{(2\pi)^{\mathsf{d}}} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} + \mathrm{i}(p^2\delta_{Z_\phi} - \delta_m) \\ &\overset{(4.1.10)}{=} \frac{-\mathrm{i}\lambda}{2} \frac{m^{\mathsf{d}-2}}{(4\pi)^{\mathsf{d}/2}} \Gamma(1 - \mathsf{d}/2) + \mathrm{i}(p^2\delta_{Z_\phi} - \delta_m)\,. \end{aligned} \tag{4.2.8}$$

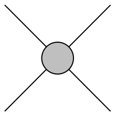
Since the first term does not depend on the momentum p^2, we obtain

$$\delta_{Z_\phi} = 0\,, \qquad \delta_m = -\frac{\lambda}{2} \frac{m^{\mathsf{d}-2}}{(4\pi)^{\mathsf{d}/2}} \Gamma(1 - \mathsf{d}/2)\,. \tag{4.2.9}$$

Namely, the divergence of the loop integral is cancelled by the counter term.

4.2.3 Four-point function

We compute the four-point function at the one-loop level. In addition to the contributions (4.1.21), we then have the counter term contribution (4.2.6b). The total contribution to the four-point function at this level is given by



$$= \quad -\mathrm{i}\lambda + \mathrm{i}\lambda^2 \sum_{x=s,t,u} V_4(x) - \mathrm{i}\delta_\lambda + \cdots . \tag{4.2.10}$$

Putting $(s,t,u) = (4m^2, 0, 0)$, we obtain

$$\delta_\lambda = \lambda^2 \left[V_4(4m^2) + 2V_4(0) \right] , \tag{4.2.11}$$

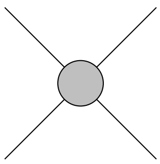
which absorbs the divergence of the vertex function.

4.3 Renormalization group

We have imposed the renormalization condition with the physical mass (4.2.7). This is very natural, but not unique way to subtract the divergences. In fact, since the condition (4.2.7) becomes singular for the massless case $m = 0$, we should modify the condition. Hence, we now impose another renormalization condition,

$$\Sigma(p^2)\Big|_{p^2=-\mu^2} = 0 , \qquad \frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}\Bigg|_{p^2=-\mu^2} = 0 , \tag{4.3.1}$$

where the parameter μ is called the **renormalization scale**. The reason why we put $p^2 = -\mu^2 < 0$ is to avoid the singularity appearing at $p^2 \geq 4m^2$. We impose similarly to (4.2.5), but more symmetric condition for the four-point function,



$$= \quad -\mathrm{i}\lambda \quad \text{at} \quad p_i^2 = -\mu^2 , \quad i = 1, \ldots, 4 . \tag{4.3.2}$$

Since the renormalization scale μ can be an arbitrary parameter, the bare correlation function does not depend on it.

We consider the connected n-point function evaluated with the renormalized coupling λ, which depends on μ. Recalling the scaling relation (4.2.2), we have the relation to the bare correlation function,

$$G_n(x_1, \ldots, x_n; \lambda, \mu) := \langle \mathsf{T}[\phi(x_1) \cdots \phi(x_n)] \rangle_\mathrm{c} = Z_\phi^{-n/2} \langle \mathsf{T}[\phi_0(x_1) \cdots \phi_0(x_n)] \rangle_{\mathrm{c},\Lambda} . \tag{4.3.3}$$

Since the bare correlation function contains infinities, we impose the cutoff Λ to regularize it. Furthermore, the bare correlation function does not depend on μ, so that its dependence appears only through Z_ϕ on the right hand side. The independence of the bare correlation function on the renormalization scale (namely, under fixing λ_0 and Λ) gives rise to the **renormalization group (RG) equation**

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda) \right] G_n(x_1, \ldots, x_n; \lambda, \mu) = 0 \tag{4.3.4}$$

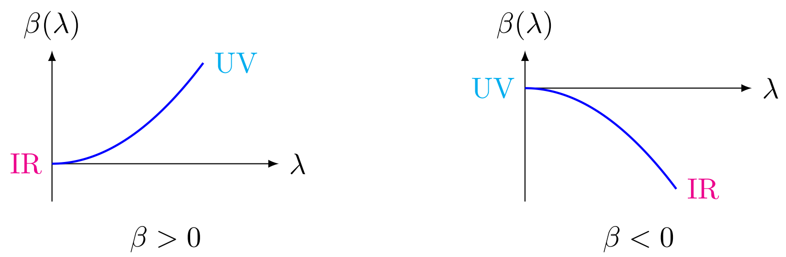
where we define the **beta function** and the **anomalous dimension**,

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu}\bigg|_{\lambda_0,\Lambda} = \frac{\partial \lambda}{\partial \log \mu}\bigg|_{\lambda_0,\Lambda} , \qquad \gamma(\lambda) = \frac{1}{2}\mu \frac{\partial \log Z_\phi}{\partial \mu}\bigg|_{\lambda_0,\Lambda} = \frac{1}{2}\frac{\partial \log Z_\phi}{\partial \log \mu}\bigg|_{\lambda_0,\Lambda} . \tag{4.3.5}$$

The beta function provides important information about the coupling constant, while the anomalous dimension describes the deviation of the operator dimension from the classical value, which is called the canonical dimension. In particular, at the zero of the beta function, $\beta(\lambda_*) = 0$, the coupling does not depend on the renormalization scale μ, which implies that the theory becomes scale invariant. Hence, the zero λ_* is called the **fixed point** of the RG flow. Such a scale invariant theory also exhibits conformal symmetry (in many cases), so that it is described as conformal field theory (CFT). See (2.3.33).

Exercise 4.4. *Verify that $\lambda = 0$ is a fixed point. (In particular, the beta function starts with an $O(\lambda^2)$ term.)*

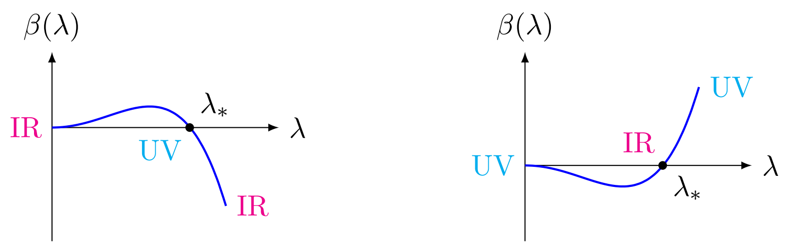
Let us discuss several situations for the beta function. The first is the beta function monotonically increasing/decreasing,



(4.3.6)

Since the renormalization scale μ characterizes the energy scale of the system, $\mu \gg 1$ and $\mu \ll 1$ correspond to the **ultraviolet (UV)** and **infrared (IR)** regimes, respectively. Hence, for $\beta > 0$, the IR fixed point theory is a free CFT. For $\beta < 0$, on the other hand, it becomes a strongly/weakly coupled theory in the IR/UV regime, which is known as the **asymptotic free**. Typical examples for the asymptotic free theory are non-Abelian gauge theory in $\mathsf{d} = 4$ and non-linear sigma model in $\mathsf{d} = 2$.

If there is a fixed point at $\lambda_* \neq 0$, it behaves as follows:



(4.3.7)

Depending on the slop of the beta function at the fixed point, we obtain non-trivial CFTs in the UV/IR limit.

Chapter 5

Quantization of non-Abelian gauge theory

In Sec. 2.6.4, we have introduced the vector field transforming under the simple Lie group G. In this Chapter, we discuss how to quantize G-gauge theory based on path integral formalism.

5.1 Gauge fixing

We consider the path integral quantization of the gauge theory with the YM Lagrangian (2.6.42),

$$\int DA \exp\left[\mathrm{i}\int \mathrm{d}x\, \mathscr{L}_{\mathrm{YM}}(A)\right]. \tag{5.1.1}$$

This naively defined path integral is problematic in the following sense: As discussed in Sec. 2.6.4, the YM Lagrangian is invariant under the gauge transformation (2.6.31). Hence, the path integral measure DA over counts the configuration, and we should instead consider that for the conjugacy class $DA_c = DA/DU$, where DU is a product of the Haar measure over G at each spacetime point x. In order to deal with this issue, we may use the **gauge fixing** trick: We restrict the gauge field configuration to that satisfies the gauge fixing condition, e.g., Coulomb gauge $\nabla \cdot A^a = 0$, Lorentz gauge $\partial^\mu A^a_\mu = 0$.

Exercise 5.1.

1. *Verify that*

$$\int \mathrm{d}x\, \delta(f(x)-c) = \int \frac{\mathrm{d}f}{f'}\delta(f-c) = \frac{1}{f'(x_*)} \qquad s.t. \qquad f(x_*) = c\,. \tag{5.1.2}$$

2. *Show the n-dimensional generalization,*

$$\int \mathrm{d}^n x\, \delta^{(n)}\left(\vec{f}(\vec{x}) - \vec{c}\right) = \det_{1\le i,j\le n}\left(\frac{\partial f_i(\vec{x}_*)}{\partial x_j}\right)^{-1} \qquad s.t. \qquad \vec{f}(\vec{x}_*) = \vec{c}\,. \tag{5.1.3}$$

We consider the functional generalization of these formulas,

$$1 = \int D\theta\, \delta(\mathsf{f}(A)) \det\left(\frac{\delta \mathsf{f}(A)}{\delta\theta}\right) \tag{5.1.4}$$

where $\theta = (\theta^a)_{a=1,\ldots,\dim G}$ parametrizes the group element $U \in G$ as in (2.6.32), and $D\theta$ is essentially the Haar measure DU in the θ-coordinate. We set $\mathsf{f}(A) = \partial^\mu A_\mu^a - f^a$, which imposes the generalized Lorentz gauge, $\partial^\mu A_\mu^a = f^a$. From the infinitesimal gauge transformation (2.6.38b) in the vicinity of the configuration $\mathsf{f}(A) = 0$, we obtain

$$\det\left(\frac{\delta \mathsf{f}(A)}{\delta\theta}\right) = \det\left(\partial^\mu D_\mu\right), \tag{5.1.5}$$

which is called the **Faddeev–Popov (FP) determinant**. We remark that this functional determinant is evaluated with the gauge field configuration satisfying the gauge fixing condition $\mathsf{f}(A) = 0$.

Inserting the identity (5.1.4), we obtain the gauge theory path integral,

$$Z = \int DA_c D\theta\, e^{iS_{\mathrm{YM}}[A]} \delta(\mathsf{f}(A)) \det\left(\partial^\mu D_\mu\right) = \int DA\, e^{iS_{\mathrm{YM}}[A]} \delta(\mathsf{f}(A)) \det\left(\partial^\mu D_\mu\right). \tag{5.1.6}$$

Here it does not depend on the parameter f^a. Then, we further insert the following identity operator to care about the delta function factor,

$$1 = \int Df \exp\left[i \int dx \left(-\frac{1}{2\xi} f^a(x) f^a(x)\right)\right], \tag{5.1.7}$$

such that

$$\begin{aligned} Z &= \int DADf\, \delta\big(\partial^\mu A_\mu^a - f^a\big) \det\left(\partial^\mu D_\mu\right) \exp\left[i \int dx \left(\mathscr{L}_{\mathrm{YM}}(A) - \frac{1}{2\xi} f^a(x) f^a(x)\right)\right] \\ &= \int DA\, \det\left(\partial^\mu D_\mu\right) \exp\left[i \int dx \left(\mathscr{L}_{\mathrm{YM}}(A) - \frac{1}{2\xi}(\partial^\mu A_\mu(x))^2\right)\right]. \end{aligned} \tag{5.1.8}$$

Such a prescription by adding the gauge symmetry breaking term to fix the gauge degrees of freedom is called the method of R_ξ **gauge**. In particular, the case with $\xi = 1$ is called the Feynman gauge, and $\xi = 0$ is called the Landau gauge. Furthermore, there is another equivalent description without explicitly introducing the gauge breaking term. Using a modification of the identity (5.1.7),

$$1 = \int DBDf \exp\left[i \int dx \left(B^a(x) f^a + \frac{\xi}{2} B^a(x) B^a(x)\right)\right], \tag{5.1.9}$$

the path integral becomes

$$Z = \int DADB\, \det\left(\partial^\mu D_\mu\right) \exp\left[i \int dx \left(\mathscr{L}_{\mathrm{YM}}(A) + B^a \partial^\mu A_\mu^a + \frac{\xi}{2} B^a B^a\right)\right]. \tag{5.1.10}$$

The auxiliary field B^a introduced here is called the **Nakanishi–Lautrup field**. The remaining part to be managed is the FP determinant in the path integral. Using the functional analog of the Gaussian integral of the Grassmann variables (3.3.7), we may write it as follows,

$$\det\left(\partial^\mu D_\mu\right) = \int DcD\bar{c} \exp\left[i \int dx \left(i\bar{c}^a \partial^\mu D_\mu c^a(x)\right)\right]. \tag{5.1.11}$$

These Grassmann scalar fields, c^a and $\bar{c}^a$, are called the **FP ghosts**, which in fact violate the spin-statistics theorem (spin-0 fermionic fields), and therefore it may violate the unitarity as well. Combining all these contributions, the path integral is now written in the following form,

$$Z = \int DADBDcD\bar{c} \exp\left[i \int dx\, \mathscr{L}_{\mathrm{tot}}(A, B, c, \bar{c})\right] \tag{5.1.12}$$

where

$$\mathscr{L}_{\text{tot}}(A,B,c,\bar{c}) = \mathscr{L}_{\text{YM}}(A) + \mathscr{L}_{\text{GF}}(A,B) + \mathscr{L}_{\text{FP}}(A,c,\bar{c})\,, \tag{5.1.13a}$$

$$\mathscr{L}_{\text{GF}}(A,B) = B^a \partial^\mu A^a_\mu + \frac{\xi}{2} B^a B^a\,, \tag{5.1.13b}$$

$$\mathscr{L}_{\text{FP}}(A,c,\bar{c}) = \mathrm{i}\bar{c}^a \partial^\mu D_\mu c^a(x) = -\mathrm{i}\partial^\mu \bar{c}^a D_\mu c^a(x)\,. \tag{5.1.13c}$$

Since there is no derivative term for the auxiliary field, the equation of motion simply yields,

$$\frac{\partial \mathscr{L}_{\text{GF}}}{\partial B^a} = \partial^\mu A^a_\mu + \xi B^a \implies B^a = -\xi^{-1}\partial^\mu A^a_\mu$$
$$\implies \mathscr{L}_{\text{GF}}\Big|_{\text{on-shell}} = -\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2\,, \tag{5.1.14}$$

which reproduces the expression (5.1.8).

5.2 BRST formalism

Apparently, there is no gauge symmetry after gauge fixing. However, instead of the local gauge symmetry, one can instead discuss the global symmetry, called the **BRST symmetry**, which is interpreted as a remnant of the original gauge symmetry.

5.2.1 BRST transforms

We replace the local gauge transform parameter with the ghost field:

$$\theta^a(x) \longrightarrow \epsilon c^a(x)\,. \tag{5.2.1}$$

Since $c^a(x)$ is a fermionic field, the transformation parameter ϵ must be also a Grassmann variable. Then, we define the **BRST transformations** by replacing the gauge transformations (2.6.38) as follows:[1]

$$\delta_{\text{B}}\phi_i(x) = \mathrm{i}g c^a(x) (t^a)^j{}_i \phi_j(x)\,, \tag{5.2.2a}$$

$$\delta_{\text{B}} A^a_\mu(x) = \Big(\partial_\mu c^a(x) - g f^a{}_{bc} A^b_\mu(x) c^c(x)\Big) = D_\mu c^a(x) \tag{5.2.2b}$$

where we shift $\theta^a(x) \to g\theta^a(x)$ from the convention of (2.6.38), and we call δ_{B} the **BRST operator**. Let $C(x) = c^a(x) t^a$. Then, we instead have the matrix version of the transformations,

$$\delta_{\text{B}}\phi(x) = \mathrm{i}g C(x)\phi(x)\,, \tag{5.2.3a}$$

$$\delta_{\text{B}} A_\mu(x) = \partial_\mu C(x) + \mathrm{i}g[A_\mu(x), C(x)]\,. \tag{5.2.3b}$$

Since δ_{B} is now a fermionic operator, it must be **nilpotent** when it acts on the fields,

$$\delta_{\text{B}}^2 = 0\,. \tag{5.2.4}$$

This property determines the transformation rule of the ghost field $c^a(x)$.

[1] In fact, δ_{B} is a spin-0 fermionic operator, which transforms physical fields (ϕ, A_μ) to ghost fields c^a. If we instead introduce a spin-$\frac{1}{2}$ fermionic operator, it transforms physical bosonic fields to physical fermionic fields $(\psi, \bar{\psi})$, and vice versa. Such a symmetry between physical bosonic and fermionic fields is called the **supersymmetry**. See, for example, [WB92, Wei95] for details.

Exercise 5.2 (The BRST transform of the ghost field $c^a(x)$)**.**

1. *Verify that*

$$\delta_B^2\phi(x) = \mathrm{i}g[(\delta_B C(x))\phi(x) - C(x)(\mathrm{i}gC(x)\phi(x))]\,. \tag{5.2.5}$$

2. *Then, show that the ghost field transforms as follows:*

$$\delta_B C(x) = \mathrm{i}gC(x)C(x)\,. \tag{5.2.6}$$

3. *Derive that for the component,*

$$\delta_B c^a(x) = -\frac{g}{2} f_{bd}{}^a\, c^b(x)c^d(x)\,. \tag{5.2.7}$$

4. *Verify the nilpotency on the gauge and ghost fields,*

$$\delta_B^2 A_\mu = 0\,, \qquad \delta_B^2 C(x) = 0\,. \tag{5.2.8}$$

In addition, the anti-ghost field behaves under the BRST operator as follows:

$$\delta_\mathrm{B}\bar{c}^a(x) = \mathrm{i}B^a\,, \tag{5.2.9a}$$

$$\delta_\mathrm{B}B^a(x) = 0\,, \tag{5.2.9b}$$

which yields $\delta_\mathrm{B}^2 = 0$ again.

5.2.2 BRST charge

Since the BRST symmetry is a global symmetry of the system, we can apply Noether's theorm to construct the conserved charge as discussed in Sec. 2.3.2.

Exercise 5.3 (Noether current for the BRST symmetry)**.** *Applying the formula shown in Sec. 2.3.2, derive the Noether current associated with the BRST symmetry, that we call the* ***BRST current****,*

$$\begin{aligned}\epsilon J_B^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu^a)}\epsilon D^\nu c^a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu c^a)}\left(-\frac{1}{2}\epsilon g f_{bd}{}^a c^b c^d\right) + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{c}^a)}\mathrm{i}\epsilon B^a \\ &= \epsilon\left[B^a D^\mu c^a - (\partial^\mu B^a)c^a + \frac{\mathrm{i}}{2} g f_{bc}{}^a(\partial^\mu \bar{c}^a)c^b c^d - \partial^\mu(F_{\mu\nu}^a c^a)\right]. \end{aligned} \tag{5.2.10}$$

We remark that the BRST current J_B^μ itself is fermionic, so that the combination ϵJ_B^μ is bosonic.

Using the BRST current, we define the **BRST charge**

$$Q_\mathrm{B} = \int \mathrm{d}^D x\, J_\mathrm{B}^0\,, \tag{5.2.11}$$

which generats the BRST transformation,

$$[\mathrm{i}\epsilon Q_\mathrm{B}, \varphi(x)] = \epsilon\delta_\mathrm{B}\varphi(x)\,. \tag{5.2.12}$$

5.2.3 BRST cohomology

In the Lagrangian (5.1.13), the ghost and anti-ghost appear as a pair, so that it is invariant under the following transformation,

$$(c^a, \bar{c}^a) \longrightarrow \left(\mathrm{e}^r c^a, \mathrm{e}^{-r} \bar{c}^a\right). \tag{5.2.13}$$

Since they are real fields, this is not a phase rotation, but a scale transformation (non-compact U(1) symmetry).

Exercise 5.4 (FP ghost charge)**.**

1. *Applying the discussion in Sec. 2.3.2, derive the Noether current associated with the scale transformation* (5.2.13)*,*

$$J_c^\mu = \mathrm{i}(\bar{c}^a D^\mu c^a - (\partial^\mu \bar{c}^a) c^a). \tag{5.2.14}$$

2. *Show that the* ***FP ghost charge*** *defined*

$$Q_c = \int \mathrm{d}^D x \, J_c^0 = \mathrm{i} \int \mathrm{d}^D x \left(\bar{c}^a D^0 c^a - \left(\partial^0 \bar{c}^a\right) c^a\right), \tag{5.2.15}$$

generates the scale transformation (5.2.13)*,*

$$[\mathrm{i}Q_c, c^a(x)] = c^a(x), \qquad [\mathrm{i}Q_c, \bar{c}^a(x)] = -\bar{c}^a(x). \tag{5.2.16}$$

We may assign the **ghost number** to each field,

Ghost number	0	+1	−1
Fields	$\phi,\ A_\mu^a,\ B^a$	$c^a,\ \delta_\mathrm{B}$	$\bar{c}^a$

(5.2.17)

In fact, the ghost number operator is given by the FP ghost charge, $N_\mathrm{c} := \mathrm{i}Q_\mathrm{c}$. Due to the nilpotency of the BRST operator, we define the **BRST complex**,[2]

$$\cdots \xrightarrow{Q_\mathrm{B}} \mathcal{C}^{-1} \xrightarrow{Q_\mathrm{B}} \mathcal{C}^0 \xrightarrow{Q_\mathrm{B}} \mathcal{C}^1 \xrightarrow{Q_\mathrm{B}} \cdots \tag{5.2.20}$$

where $\mathcal{C}^p$ denotes the sector with the ghost number p. Then, we define the **BRST cohomology**

$$H_\mathrm{B} = \ker Q_\mathrm{B} / \operatorname{im} Q_\mathrm{B}, \tag{5.2.21}$$

where $\ker Q_\mathrm{B}$ and $\operatorname{im} Q_\mathrm{B}$ are given by

$$Q_\mathrm{B} \left|\psi\right\rangle = 0, \qquad \left|\psi\right\rangle \in \ker Q_\mathrm{B}, \tag{5.2.22a}$$

$$\left|\psi\right\rangle = Q_\mathrm{B} \left|\psi'\right\rangle, \qquad \left|\psi\right\rangle \in \operatorname{im} Q_\mathrm{B}. \tag{5.2.22b}$$

[2] We can also introduce the anti-BRST operator having ghost number −1 and mapping from $\mathcal{C}^p$ to $\mathcal{C}^{p-1}$ as follows:

$$\overline{Q}_\mathrm{B} = \mathrm{C}_\mathrm{FP} \, Q_\mathrm{B} \, \mathrm{C}_\mathrm{FP}^{-1}, \tag{5.2.18}$$

where we denote the FP conjugation operator by C_FP,

$$\mathrm{C}_\mathrm{FP}: \qquad c^a \longrightarrow \bar{c}^a, \qquad \bar{c}^a \longrightarrow -c^a, \qquad B^a \longrightarrow -\overline{B}^a. \tag{5.2.19}$$

Apparently, $\operatorname{im} Q_{\mathrm{B}} \subset \ker Q_{\mathrm{B}}$. This is why one may consider their quotient to define the cohomology. We remark that any state in $\operatorname{im} Q_{\mathrm{B}}$ written as $|\psi\rangle = Q_{\mathrm{B}} |\psi'\rangle$ has zero norm,

$$\langle\psi|\psi\rangle = \langle\psi| Q_{\mathrm{B}} |\psi'\rangle = 0 \,. \tag{5.2.23}$$

This is an important property that ensures that unphysical states and the ghosts are not physically observed in the BRST formalism [KO79].

5.2.4 Lagrangian in the BRST formalism

Let $\mathscr{F}^a(\phi, A_\mu, B, c, \bar{c})$ be an arbitrary function of the fields with ghost number zero. Then, we add the following term to the Lagrangian

$$\mathscr{L}_{\mathrm{GF+FP}} = -\mathrm{i}\delta_{\mathrm{B}}(\bar{c}^a \mathscr{F}^a) \,. \tag{5.2.24}$$

From the nilpotency of the operator $\delta_{\mathrm{B}}^2 = 0$, we see that it is BRST invariant,

$$\delta_{\mathrm{B}}\mathscr{L}_{\mathrm{GF+FP}} = -\mathrm{i}\delta_{\mathrm{B}}^2(\bar{c}^a \mathscr{F}^a) = 0 \,. \tag{5.2.25}$$

Recalling the original YM Lagrangian is gauge invariant, the total Lagrangian $\mathscr{L}_{\mathrm{tot}}$ is also BRST invariant, $\delta_{\mathrm{B}}\mathscr{L}_{\mathrm{tot}} = 0$. Therefore, although the original gauge symmetry is broken due to the additional term $\mathscr{L}_{\mathrm{GF+FP}}$, the current system instead has the BRST symmetry.

Exercise 5.5 (Gauge fixing and FP terms)**.**

1. *Show that, if $\mathscr{F}^a$ does not contain the ghost and anti-ghost fields, it gives*

$$\mathscr{L}_{GF+FP} = B^a \mathscr{F}^a + \mathrm{i}\bar{c}^a(\delta_B \mathscr{F}^a) =: \mathscr{L}_{GF} + \mathscr{L}_{FP} \,. \tag{5.2.26}$$

2. *Put*

$$\mathscr{F}^a = \partial^\mu A_\mu^a + \frac{\xi}{2} B^a \,. \tag{5.2.27}$$

Then, verify that it reproduces the Lagrangian (5.1.13)*.*

If $\mathscr{F}^a$ contains the ghost fields, there will be non-linear terms of c^a and $\bar{c}^a$ in the Lagrangian $\mathscr{L}_{\mathrm{GF+FP}}$, so that we cannot integrate out the ghosts to obtain the FP determinant. In this sense, the BRST formalism is a genaralization of the FP gauge fixing formalism.

5.2.5 Feynman rule

Based on the BRST formalism discussed above, let us consider the Feynman rule to deal with the petrurbative computation of the amplitudes. In addition to the Lagrangian (5.1.13), we also take into account the Dirac field coupled with the gauge field,

$$\mathscr{L}_{\mathrm{Dirac}} = \bar{\psi}[\gamma^\mu(\mathrm{i}\partial_\mu + gA_\mu) - m]\psi = \bar{\psi}[\mathrm{i}\gamma^\mu D_\mu - m]\psi \,, \tag{5.2.28}$$

where $D_\mu = \partial_\mu - \mathrm{i}gA_\mu$ is the covariant derivative acting on a field in the fundamental representation of G.

Exercise 5.6.

1. *Verify that the quadratic (free) part of the Lagrangian is given by*

$$\mathscr{L}_0 = \frac{1}{2} A^{a\mu} \left[g_{\mu\nu} \partial_\rho \partial^\rho - \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] A^{a\nu} + \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi + \bar{c}^a \mathrm{i} \partial^\mu \partial_\mu c^a \tag{5.2.29}$$

where the gauge fixing parameter ξ is interpreted as the renormalized one.

2. *From the YM Lagrangian (2.6.42), derive the self-interaction terms of the gauge fields,*

$$\mathscr{L}_{\text{3-gauge}} = -\frac{g}{2} f_{bc}{}^{a} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} \,, \tag{5.2.30a}$$

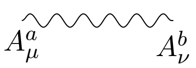
$$\mathscr{L}_{\text{4-gauge}} = -\frac{g^2}{4} f_{bc}{}^{a} f_{de}{}^{a} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \,. \tag{5.2.30b}$$

3. *Derive the gauge-Dirac and gauge-ghost interaction terms,*

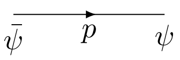
$$\mathscr{L}_{\text{gauge-Dirac}} = g \bar{\psi}^i \gamma_\mu A^{a\mu} (t^a)_i{}^j \psi_j \,, \tag{5.2.31a}$$

$$\mathscr{L}_{\text{gauge-ghost}} = -\mathrm{i} g f^a{}_{bc} \bar{c}^a \partial^\mu A^b_\mu c^c = \mathrm{i} g f^a{}_{bc} (\partial^\mu \bar{c}^a) A^b_\mu c^c \,. \tag{5.2.31b}$$

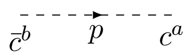
From these expressions, we obtain the following Feynman rules for the propagators,

Gauge propagator: 

$$= \frac{\mathrm{i}\delta^{ab}}{p^2 + \mathrm{i}\varepsilon} \left[g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right] \tag{5.2.32a}$$

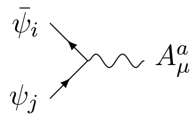
Dirac propagator: 

$$= \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon} \tag{5.2.32b}$$

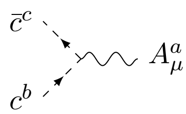
Ghost propagator: 

$$= \frac{\mathrm{i}\delta^{ab}}{\mathrm{i}(p^2 + \mathrm{i}\varepsilon)} \tag{5.2.32c}$$

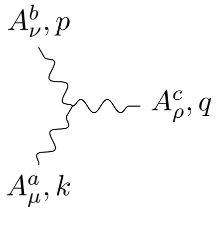
and for the vertices,

Dirac-gauge vertex: 

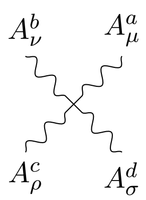
$$= \mathrm{i} g \gamma_\mu (t^a)_i{}^j \tag{5.2.33a}$$

Ghost-gauge vertex: 

$$= \mathrm{i} g f_{bc}{}^{a} p_\mu \tag{5.2.33b}$$

3-gauge: 

$$= -g f_{abc} [g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\mu}(q-k)_\nu] \tag{5.2.33c}$$

4-gauge: 

$$\begin{aligned} = \; & -\mathrm{i} g f_{abe} f_{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \\ & - \mathrm{i} g f_{ace} f_{bde} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\rho\nu}) \\ & - \mathrm{i} g f_{ade} f_{bce} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu}) \end{aligned} \tag{5.2.33d}$$

Chapter 6

Spontaneous symmetry breaking

Let $E(p)$ be the spectrum of the system with momentum p. The system with $E(p=0)=0$ is called **gapless**, while it is called **gapped** if $E(p=0)\neq 0$. For example, the spectrum of the free scalar field is given by $E(p)=\sqrt{p^2+m^2}$, so that it is gapped unless $m=0$. In fact, (almost) all the gapless systems are associated with the **spontaneous symmetry breaking (SSB)**, which is one of the most important concepts in QFT.

6.1 Nambu–Goldstone's theorem

Suppose that Q is the conserved (Noether) charge associated with the continuous symmetry introduced in Sec. 2.3.2. We denote the vacuum/ground state by $|0\rangle$. Then, there are two possibilities,

$$\text{(i)}\quad Q\,|0\rangle = 0\,, \qquad \text{(ii)}\quad Q\,|0\rangle \neq 0\,. \tag{6.1.1}$$

Since Q is the generator of the symmetry transformation, $|0\rangle$ is invariant under it in the case (i), while it is not invariant in the case (ii). The case (ii) actually shows the SSB: the vacuum state violates the symmetry of the Lagrangian.

Recalling the definition of the conserved charge (2.3.8), defined as the spatial integral, its norm $\langle 0|\,QQ\,|0\rangle$ shall be proportional to the spatial volume V, if it exists. Since it diverges in the thermodynamic limit $V\to\infty$, the case (ii) would be ill-defined (its normalization is indeterminate). Hence, we instead consider the local version of the case (ii) as follows,

$$\text{(ii')}\quad \langle\delta\varphi\rangle := \langle 0|\,\delta\varphi(x)\,|0\rangle \overset{(2.3.11)}{=} \langle 0|\,[\mathrm{i}Q,\varphi(x)]\,|0\rangle \neq 0\,. \tag{6.1.2}$$

The vacuum expectation value (vev) of the local field operator $\delta\varphi$ denoted by $\langle\delta\varphi\rangle$ is called the **order parameter**.

Suppose that the field operator transforms in the representation R of the symmetry group G, $\delta\varphi_i = \mathrm{i}(t^a)_{ij}\varphi_j$, where $(t^a)_{a=1,\ldots,\dim G}$ is a set of the generators of G, and denote the associated conserved charge by Q^a. Consider the potential for $\varphi(x)$ denoted by $V(\varphi)$,[1] which is invariant under the infinitesimal transformation,

$$0 = V(\varphi+\epsilon\delta\varphi) - V(\varphi) = \mathrm{i}\epsilon\frac{\partial V(\varphi)}{\partial\varphi_j}(t^a)_{jk}\varphi_k + O(\epsilon^2)\,. \tag{6.1.3}$$

[1] This potential should be understood as the effective action discussed in Sec. 3.6.

Taking the derivative with φ_i again, we have

$$\frac{\partial^2 V(\varphi)}{\partial\varphi_i \partial\varphi_j}(t^a)_{jk}\varphi_k + \frac{\partial V(\varphi)}{\partial\varphi_j}(t^a)_{jk}\delta_{ik} = 0\,. \tag{6.1.4}$$

We denote the vev of the field by $\bar{\varphi} = \langle\varphi\rangle$, which obeys $\partial V/\partial\varphi|_{\varphi=\bar{\varphi}} = 0$. Then, we obtain

$$M_{ij}(t^a)_{jk}\bar{\varphi}_k = 0 \quad\Longleftrightarrow\quad M_{ij}\,\langle\delta\varphi_j\rangle = 0\,, \tag{6.1.5}$$

where we define the mass matrix,

$$M_{ij} = \left.\frac{\partial^2 V(\varphi)}{\partial\varphi_i\partial\varphi_j}\right|_{\varphi=\bar{\varphi}}\,. \tag{6.1.6}$$

Namely, the order parameter turns out to be the zero mode (eigenvector of the zero eigenvalue) associated with the mass matrix if it is nonzero. Let H be a subgroup of G, s.t.,

$$\langle\delta\varphi\rangle = \langle\mathrm{i}(t^a)\varphi\rangle \begin{cases} \neq 0 & (t^a \in \operatorname{Lie} G/H) \\ = 0 & (t^a \in \operatorname{Lie} H) \end{cases} \tag{6.1.7}$$

This shows that the symmetry is broken from G to H, and the field transforming under the quotient G/H behaves as the zero mode, called the **Nambu–Goldstone (NG) mode**. We denote the number of broken symmetries by $N_{\mathrm{BS}} = \dim G/H = \dim G - \dim H$. Then, the **Nambu–Goldstone's theorem** claims that it agrees with the number of the NG modes in the relativistic (Lorentz symmetric) system,

$$N_{\mathrm{NG}} = N_{\mathrm{BS}}\,. \tag{6.1.8}$$

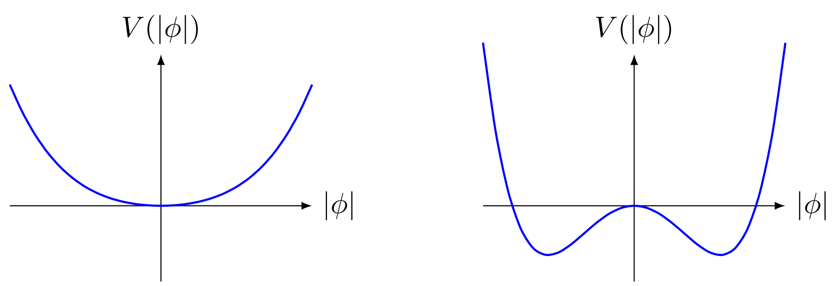
If there is no Lorentz symmetry, this equality does not hold in general, $N_{\mathrm{NG}} \leq N_{\mathrm{BS}}$. See, for example, [Wat19] for details in this case.

Complex scalar field

We consider the complex scalar field discussed in Sec 2.4.2 as a primary example which shows the SSB of the global symmetry. The Lagrangian is given in (2.4.5). Due to the U(1) symmetry (2.4.6), the potential only depends on the absolute value of the field,

$$V(|\phi|) = m^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4\,. \tag{6.1.9}$$

We assume $\lambda > 0$ for the finiteness of the action. Then, there are two possibilities depending on the mass parameter:



(a) $m^2 > 0$ (b) $m^2 < 0$

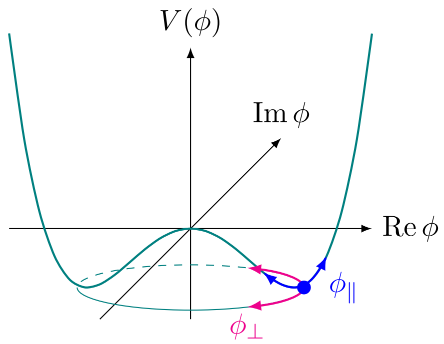
(6.1.10)

The vacuum is given by the stationary point of the potential, $\partial V(|\phi|)/\partial\phi = 0$. In the case (a), $\phi = 0$ is the unique solution. On the other hand, in the case (b),[2] there are two stationary points, $|\phi| = 0$ and $\sqrt{-m^2/\lambda} =: v/\sqrt{2}$. As seen in the potential, the former one seems unstable against the fluctuation. In fact, the negative squared mass (imaginary mass) indicates instability of the vacuum, which is called the tachyon. Any points on the circle $|\phi| = \sqrt{-m^2/\lambda}$ are equivalent, but once specifying the phase of the field, the U(1) symmetry is spontaneously broken in the vacuum.

We take $\phi = v/\sqrt{2}$ as a vacuum configuration, and we expand the field around this vacuum,

$$\phi(x) = \frac{1}{\sqrt{2}}\big(v + \phi_{\parallel}(x) + \mathrm{i}\phi_{\perp}(x)\big)\,. \tag{6.1.11}$$

The real fields, $\phi_{\parallel}(x)$ and $\phi_{\perp}(x)$, describe the fluctuations along the parallel (real) and perpendicular (imaginary) directions as follows:



(6.1.12)

Exercise 6.1. *Plugging in the expansion* (6.1.11)*, derive the Lagrangian for the fields* $(\phi_{\parallel}(x), \phi_{\perp}(x))$*,*

$$\begin{aligned}\mathscr{L}(\phi_{\parallel}, \phi_{\perp}) = &\frac{1}{2}(\partial_\mu\phi_{\parallel})^2 - \frac{1}{2}\tilde{m}^2\phi_{\parallel}^2 + \frac{1}{2}(\partial_\mu\phi_{\perp})^2 \\ &- \tilde{m}\sqrt{\lambda}\phi_{\parallel}\left(\phi_{\parallel}^2 + \phi_{\perp}^2\right) - \frac{\lambda}{8}\left(\phi_{\parallel}^2 + \phi_{\perp}^2\right)^2 - V(v/\sqrt{2})\,,\end{aligned} \tag{6.1.13}$$

where $\tilde{m}^2 = -2m^2 > 0$.

This shows that $\phi_{\parallel}$ has the positive squared mass, while $\phi_{\perp}$ is a massless field, which is identified with the NG mode associated with the SSB of the global U(1) symmetry.

6.2 Higgs mechanism

So far, we have discussed the spontaneous breaking of global symmetry. One can similarly consider it in the system with the local gauge symmetry.[3] In that case, the NG mode can be coupled with the gauge field and giving a mass. Such a mechanism to generate the massive gauge field is called the **Higgs mechanism**.

[2]The complex scalar field model with $m^2 < 0$ is called the **Goldstone model**, which shows the spontaneous breaking of the global U(1) symmetry as shown below.

[3]This is not the spontaneous breaking of local gauge symmetry. It is known as **Elitzur's theorem** that local gauge symmetries cannot be spontaneously broken.

6.2.1 Abelian Higgs model

We consider the complex scalar theory coupled with U(1) gauge field,

$$\mathscr{L}(\phi, A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi(x)|^2 - V(|\phi|) \tag{6.2.1}$$

where the potential $V(|\phi|)$ is given by (6.1.9). See Sec. 2.6 for the definition of the covariant derivative and the Maxwell term. This model is called the **Abelian Higgs model** and also the **scalar QED**.

We now consider the case (b) of the potential $V(|\phi|)$ with $\mu^2 = -m^2 > 0$. Then, we have the same vev of the scalar field as before, $\phi = v/\sqrt{2} = \sqrt{\mu^2/\lambda}$. We again expand the field around this configuration, but with a slightly different parametrization,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \sigma(x))\mathrm{e}^{-\mathrm{i}\pi(x)}\,. \tag{6.2.2}$$

Roughly speaking, they correspond to the previous parametrization as $(\sigma, \pi) \sim (\phi_\parallel, \phi_\perp)$. Then, applying the gauge transformation (2.6.1), $\phi(x) \to \mathrm{e}^{\mathrm{i}\pi(x)}\phi(x) = (v + \sigma(x))/\sqrt{2}$, the field $\pi(x)$ does not appear in the Lagrangian any longer. This gauge choice is called the unitary gauge.

Exercise 6.2. *Plug in the expression of the scalar field, $\phi(x) = (v + \sigma(x))/\sqrt{2}$, obtain the Lagrangian in terms of the new scalar field $\sigma(x)$ and the gauge field A_μ,*

$$\mathscr{L}(\sigma, A_\mu) = \mathscr{L}_{gauge}(A_\mu) + \mathscr{L}_\sigma(\sigma) + \mathscr{L}_{int}(\sigma, A_\mu) - V(v/\sqrt{2})\,, \tag{6.2.3a}$$

$$\mathscr{L}_{gauge}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{(ev)^2}{2}A_\mu A^\mu\,, \tag{6.2.3b}$$

$$\mathscr{L}_\sigma(\sigma) = \frac{1}{2}(\partial_\mu\sigma(x))^2 - \frac{1}{2}\tilde{m}^2\sigma^2 - \frac{\lambda v}{2}\sigma^3 - \frac{\lambda}{8}\sigma^4\,, \tag{6.2.3c}$$

$$\mathscr{L}_{int}(\sigma, A_\mu) = e^2\left(v\sigma + \frac{1}{2}\sigma^2\right), \tag{6.2.3d}$$

where $\tilde{m}^2 = -2m^2 = 2\mu^2 > 0$ as before.

In this expression, we find the mass term for the gauge field, $A_\mu A^\mu$, which breaks the gauge symmetry. In addition, in contrast to the Goldstone model, there is no gapless NG mode, but only a massive scalar field $\sigma(x)$. In fact, originally the scalar field $\pi(x)$ is supposed to play a role of the NG mode, which is then absorbed by the gauge field via the U(1) transformation (2.6.4), $A_\mu \to A_\mu + e^{-1}\partial_\mu\pi(x)$.

The vector field has spin 1, which involves three states labeled by the angular momentum in the going direction, $(+1, 0, -1)$, called the helicity. The massless vector field, e.g., the photon, is fully polarized (transverse modes only), so that its helicity must be ± 1. Now, the scalar field $\pi(x)$ is absorbed by the gauge field as a longitudinal mode (zero helicity) to give a mass for it. This mechanism to generate a mass for the gauge field is called the **Higgs mechanism**. In general, the mass for the gauge field is proportional to the vev of the scalar field.

6.2.2 Higgs–Kibble model

Let us consider the system with non-Abelian gauge symmetry and the corresponding spontaneous breaking with $G = \mathrm{SU}(2)$. Then, the generators are given by the Pauli matrices, $(t^a)_{a=1,2,3} =$

$(\sigma^a/2)_{a=1,2,3}$, and the structure constant is given by the anti-symmetric tensor, $f^{abc} = \epsilon^{abc}$. We use the vector notation for the SU(2) structure

$$A_\mu = \vec{A}_\mu \cdot \vec{t} = \sum_{a=1}^{3} A_\mu^a t^a \,, \qquad \left(\vec{X} \times \vec{Y}\right)^a = \epsilon^{abc} X_b Y_c \,, \qquad \left|\vec{X}\right|^2 = \sum_{a=1}^{3} X^a X^a \,. \tag{6.2.4}$$

In this convention, the field strength is written as

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g \vec{A}_\mu \times \vec{A}_\nu \,, \tag{6.2.5}$$

with the coupling constant g. The Higgs field forms an SU(2) doublet (**2**-representation),

$$\Phi = (\phi_1 \quad \phi_2)^{\mathrm{T}} \,, \tag{6.2.6}$$

and its covariant derivative is given by

$$D_\mu \Phi = \left(\partial_\mu - \mathrm{i} g \vec{A}_\mu \cdot \vec{t}\right) \Phi \,. \tag{6.2.7}$$

Then, we define the **SU(2) Higgs–Kibble model**,

$$\mathscr{L} = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2} \left(\Phi^\dagger \Phi\right)^2 . \tag{6.2.8}$$

The vacuum structure of this model is quite similar to the Abelian case (6.2.1): The vacuum configuration is given by $\Phi = (0 \quad v/\sqrt{2})^{\mathrm{T}}$ with $v = \sqrt{2\mu^2/\lambda}$.

Exercise 6.3. *Expanding the Higgs field around the vacuum,*

$$\Phi(x) = \frac{1}{\sqrt{2}} \left(v + \phi_\parallel(x) + \mathrm{i} \vec{\phi}_\perp \cdot \vec{t}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_\perp^2(x) + \mathrm{i}\phi_\perp^1(x) \\ v + \phi_\parallel(x) - \mathrm{i}\phi_\perp^3(x) \end{pmatrix} , \tag{6.2.9}$$

and plugging in this expression to the Lagrangian, show that it is written as

$$\begin{aligned} \mathscr{L} = & -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} + \frac{1}{2} M^2 \left| \vec{A}_\mu - M^{-1} \partial_\mu \vec{\phi}_\perp \right|^2 + \frac{1}{2} (\partial_\mu \phi_\parallel)^2 - \frac{1}{2} \tilde{m}^2 \phi_\parallel^2 \\ & + \frac{g}{2} \vec{A}_\mu \cdot \left(\vec{\phi}_\perp \partial^\mu \phi_\parallel - \phi_\parallel \partial^\mu \vec{\phi}_\perp + \vec{\phi}_\perp \times \partial^\mu \vec{\phi}_\perp \right) + \frac{g}{2} M \vec{A}_\mu \cdot \vec{A}^\mu + \frac{g^2}{8} \vec{A}_\mu \cdot \vec{A}^\mu \left(\phi_\parallel^2 + \left| \vec{\phi}_\perp \right|^2 \right) \\ & - \frac{1}{2} \tilde{m} \sqrt{\lambda} \phi_\parallel \left(\phi_\parallel^2 + \left| \vec{\phi}_\perp \right|^2 \right) - \frac{\lambda}{8} \left(\phi_\parallel^2 + \left| \vec{\phi}_\perp \right|^2 \right)^2 - V(v/\sqrt{2}) \end{aligned} \tag{6.2.10}$$

where $M = gv/2$ and $\tilde{m}^2 = 2\mu^2 > 0$.

Precisely speaking, since this is non-Abelian gauge theory, we should take care of the gauge fixing with the FP ghosts. Up to this point, the kinetic term of the perpendicular mode is removed by the gauge transformation (unitary gauge), and the gauge field becomes massive similarly to the Abelian model.

6.2.3 Weinberg–Salam model

It has been established that the weak interaction is described by $\mathrm{SU}(2)_{\mathrm{W}}$ gauge theory. It has been known that only the left-handed leptons transform under $\mathrm{SU}(2)_{\mathrm{W}}$, and thus the right-handed component is a singlet under it. We denote the electron and the corresponding neutrino (electron neutrino) by e and ν_e. Then, the left- and right-handed fields are given by (See (2.5.9))

$$L_e := \begin{pmatrix} \nu_e \\ e \end{pmatrix}_{\mathrm{L}} = \frac{1-\gamma_5}{2} \begin{pmatrix} \nu_e \\ e \end{pmatrix} , \qquad R_e := e_{\mathrm{R}} = \frac{1+\gamma_5}{2} e \,. \tag{6.2.11}$$

We have two more similar pairs for the muon (μ, ν_μ) and the tau particle (τ, ν_τ), known as the three generation structure. Denote the generators of $\mathrm{SU}(2)_\mathrm{W}$ by $(T^a)_{a=1,2,3}$. Since the left-handed lepton transforms as a doublet, the corresponding "spin" called the weak isospin is given by $\frac{1}{2}$. ν_e and e correspond to $T^3 = \pm\frac{1}{2}$.

The neutrinos are, as their names indicate, charge neutral particles described as left-handed Weyl spinors. On the other hand, the electron has charge -1 (same for μ and τ). Since the doublet does not have the same electric charge, $\mathrm{SU}(2)_\mathrm{W}$ should be mixed up with electromagnetic $\mathrm{U}(1)_\mathrm{EM}$. In order to describe this situation, we introduce the additional $\mathrm{U}(1)_Y$ symmetry, and call the corresponding charge the weak hypercharge Y. Then, the electric charge is given by the following combination,

$$Q = T^3 + Y \, . \tag{6.2.12}$$

Assigning $Y = -\frac{1}{2}$ and -1 for L_e and R_e, this explains the electric charges of ν_e and e are 0 and -1. Since the photon (electromagnetic gauge field) is the unique massless gauge field, the remaining symmetries must be spontaneously broken,

$$\mathrm{SU}(2)_\mathrm{W} \times \mathrm{U}(1)_Y \longrightarrow \mathrm{U}(1)_\mathrm{EM} \, . \tag{6.2.13}$$

This is known as the unification of the electromagnetism and the weak interaction.

We consider the **Weinberg–Salam model**, which shows the symmetry beaking of the form of (6.2.13). We denote the $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_\mathrm{W}$, and $\mathrm{U}(1)_\mathrm{EM}$ gauge fields by B_μ, $\vec{A}_\mu$, and A_μ. Their gauge couplings are g' and g. Then, the Lagrangian is given as follows,

$$\mathscr{L} = \mathscr{L}_\text{gauge} + \mathscr{L}_\text{Higgs} + \mathscr{L}_\text{lepton} \, , \tag{6.2.14a}$$

$$\mathscr{L}_\text{gauge} = -\frac{1}{4}\left|\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu\right|^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \, , \tag{6.2.14b}$$

$$\mathscr{L}_\text{Higgs} = (D_\mu \Phi)^\dagger (D^\mu \Phi) + \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2}\left(\Phi^\dagger \Phi\right)^2 , \tag{6.2.14c}$$

$$\mathscr{L}_\text{lepton} = \overline{L}_e \mathrm{i}\gamma^\mu D_\mu L_e + \overline{R}_e \mathrm{i}\gamma^\mu D_\mu R_e - f_e\left[\left(\overline{L}_e \Phi\right) R_e + \overline{R}_e\left(\Phi^\dagger L_e\right)\right] \tag{6.2.14d}$$

where f_e is the Yukawa coupling constant, and the covariant derivative is given by

$$D_\mu = \begin{cases} \partial_\mu - \mathrm{i}g' Y B_\mu - \mathrm{i}g\vec{A}_\mu \cdot \vec{t} & (L_e, \, \Phi) \\ \partial_\mu - \mathrm{i}g' Y B_\mu & (R_e) \end{cases} . \tag{6.2.15}$$

Since the Higgs sector is the same as the Kibble–Higgs model (6.2.8), we have the same vev for the Higgs field, $\Phi = \begin{pmatrix} 0 & v/\sqrt{2} \end{pmatrix}^\mathrm{T}$. We assign $Y = +\frac{1}{2}$ to the Higgs field so that the lower component has $Q = 0$, which is required to preserve the $\mathrm{U}(1)_\mathrm{EM}$ symmetry.

Exercise 6.4.

1. *Show that the mass term for the gauge field is obtained from the covariant derivative of the Higgs field,*

$$\left| -\frac{\mathrm{i}}{2} \begin{pmatrix} g' B_\mu + g A_\mu^3 & g A_\mu^1 - \mathrm{i} g A_\mu^2 \\ g A_\mu^1 + \mathrm{i} g A_\mu^2 & g' B_\mu - g A_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right|^2 = M_W^2 W_\mu^\dagger W^\mu + \frac{1}{2} M_Z^2 Z_\mu Z^\mu \tag{6.2.16}$$

where we define

$$W_\mu = \frac{1}{\sqrt{2}}(A^1_\mu - \mathrm{i}A^2_\mu)\,, \tag{6.2.17a}$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\big(gA^3_\mu - g'B_\mu\big)\,, \tag{6.2.17b}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\big(gA^3_\mu + g'B_\mu\big)\,, \tag{6.2.17c}$$

and the mass parameters are

$$M_W^2 = \frac{1}{4}g^2v^2\,, \qquad M_Z^2 = \frac{1}{4}(g^2+g'^2)v^2\,. \tag{6.2.18}$$

Verify that there is no mass term for the gauge field A_μ. *The electric charges of* W, $W^\dagger$, Z, A *are given by* $Q = +1, -1, 0, 0$. *The charged and neutral vector fields are called the W boson and Z boson.*

2. *Define* θ_W, *called the Weinberg angle, s.t.,*

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}. \tag{6.2.19}$$

Namely, $\tan\theta_W = g'/g$. *Then, show that*

$$M_Z^2 = \frac{M_W^2}{\cos\theta_W}\,. \tag{6.2.20}$$

3. *Show that the mass term for the lepton is obtained from the Yukawa interaction with non-zero vev of the Higgs field,*

$$-f_e\Big[\big(\overline{L}_e\Phi\big)R_e + \overline{R}_e\Big(\Phi^\dagger L_e\Big)\Big] = -f_e\frac{v}{\sqrt{2}}\,\bar{e}e =: m_e\,\bar{e}e\,. \tag{6.2.21}$$

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